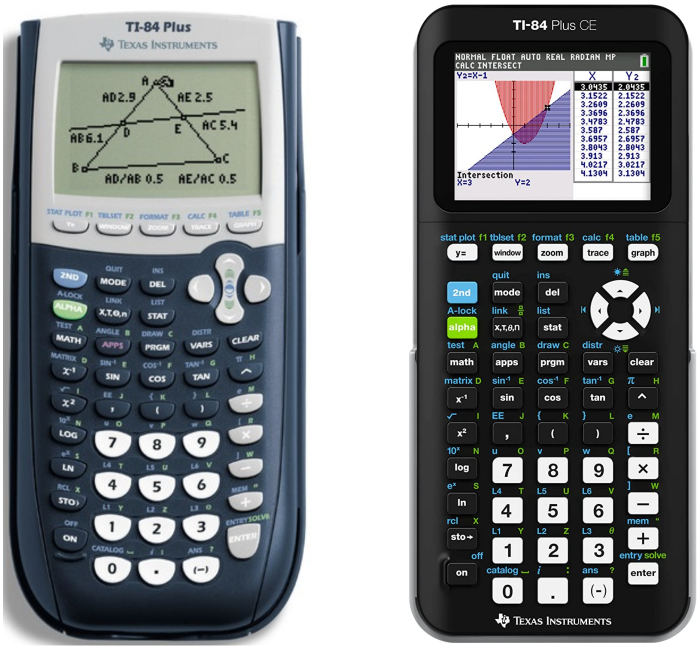
Mathematics on a TI-84/CE

Volume 1: Basics

Chapters 1– 8



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Contents

Mathematics on a TI-84/CE

This, the first of four volumes, contains material suitable for students and teachers up to about Year 10. Chapter 2, *Getting Started*, is of course suitable for anyone starting to use a TI-84/CE.

Volume 2 of this book contains topics directly relevant to Calculus and its applications, although the first chapter, *Functions and their Graphs*, is of more general relevance. The topics in Volume 2 are: Functions and their Graphs; Graph and Calculus Operations; Numerical Integration; Taylor Series; Differential Equations; Population Modelling 2 – Logistic and Epidemic Models; and Multivariable Calculus.

The last chapter in Volume 2 gives a list of the programs cited in all volumes of the book, and full information on copying and using these programs.

Volume 3 of this book contains more advanced topics, relevant to students and teachers of Specialist Mathematics and first-year university Mathematics courses. The topics in Volume 3 are: Sequences and Series; Probability and Statistics 2 – Probability Distributions and Hypothesis Testing; Matrices and Vectors; Population Modelling 3 – Matrix Models; Fitting Curves to Data 2; Financial Mathematics 2 – TVM Calculations; Complex Numbers; and Programming.

Volume 4 of this book contains 28 Mathematics labs or projects, together with a Lab Manual for teachers/instructors.

Calculator versions

Currently (early 2022), TI-84 calculators come in two versions: the TI-84Plus and the more recent TI-84CE. The main difference is that the CE screen has much higher resolution. It also has colour but I have done most of the screens in black and white to avoid the need for colour printers or photocopiers. Calculations, screenshots and figures were done on a TI-84CE in CLASSIC mode.

Some programs have had to be changed for the CE because of the different screen: I usually append 'CE' to the program name to indicate this. All the programs here are available at *www.XXX*.

1 Graphics Calculators and Mathematics

1.1 Introduction

Mathematics is a visual subject, and graphics calculators can provide the picture in a number of important areas of Mathematics. They are also useful in allowing students to explore mathematics numerically and graphically, and to do realistic mathematical modelling, asking *what if* questions of a model. In fact, any modelling that does not use some sort of technology to do calculations quickly soon becomes very boring. Graphics calculators are portable, powerful and, for what they do, relatively cheap.

1.2 What can graphics calculators do?

- All the features of a scientific calculator plus matrices, statistics, probability and complex numbers. See the various topics in the three volumes of this book.
- Multi-line screen, which displays input and output of calculations simultaneously.
- Recall and editing of previous entries and answers.
- Ability to plot Cartesian, parametric, polar and sequence graphs, and tables of function values.
- Graph/Calculus (numerical) operations for finding zeros, maxima and minima, and intersections of functions, derivatives at a specified point and definite integrals.
- Statistical functions for organising, analysing and displaying data; probability distributions.
- Can be linked to other calculators, computers and printers for electronic transfer of programs, data, etc and downloading programs from a computer or the web.
- Programmable, with a large number of programs available for downloading.[1]
- Can be used in conjunction with a calculator-based data logger: this enables easy collection of real data, which can be organised, displayed and analysed on the calculator.
- An emulator for computers is available.

At a more mundane level, graphics calculators are fun. Students pick up the operations very quickly (much faster than teachers), and if you can't get your students to use a graphics calculator, there are heaps of games on the web to tempt them.

Getting started is always the hardest, especially when you have to modify or write new courses, but the experience at UNSW Canberra and most other schools and universities at which graphics calculators have been used for a while, is that graphics calculators should not just be an add-on to a course, but should be integrated fully, including their use in tests and exams. They should enhance student understanding, not replace it.[2] This raises some issues,

[1] In my experience, programming a TI-84 calculator is the easiest way to start coding. The version of Visual Basic in the calculator provides the ability to program all the calculator functions.

[2] This is one reason I much prefer to use the TI-84 rather than the TI-nSpire, which does symbolic manipulation (CAS) and has large numbers of 'black-box' commands in a whole network of menus.

most of which are resolvable. You might like to read for example *Graphics calculators in the mathematics curriculum: Integration or differentiation?* by Jen Bradley, Barry Kissane and Marian Kemp about their experiences in WA.[3]

1.3 Implications for teaching and learning

- A need to think about classroom dynamics.
- Improved student motivation.
- Enhanced modelling and exploration opportunities.
- The potential for using an 'animated' whiteboard.[4]

The graphics calculator is a tool that can assist teachers, but there is a need to think about its use in the classroom. We need to take care that we don't hand over all of our teaching to the technology. The technology needs to be used to enhance students' understanding, not replace it. It provides a valuable tool for drawing links between various content strands and to complement traditional tools such as pencil and paper.

It is important that teachers remain in control of the learning environment, but the classroom dynamics change — there is more exploration and a problem-solving approach to learning can be encouraged. This may require a change in teaching methodology, the teacher becoming a facilitator of student learning by the use of a wider variety of teaching strategies. The whole process needs to be approached with careful thought, as well as a detemination to persevere if early problems arise.

The use of graphics calculators certainly motivates students. It provides them with different ways (graphical and numerical) of looking at mathematics, is less tedious for a number of necessary tasks once the basics have been learnt, promotes student investigation by allowing them to explore concepts independently and enhances modelling opportunities by doing the basic calculations quickly, perhaps using a program.

1.4 Using graphics calculators in modelling

Modelling with mathematics is problem-solving using practical examples, preferably ones that students have some familiarity with. The importance of problem-solving in Mathematics education was well put by Thelma Perso:[5]

> For too long we have concentrated on teaching students the 'bits' and the 'tools' for applying and solving mathematical problems but have paid little, if any, attention to teaching children how to use them. Someone once said that if we taught English like we teach Mathematics, children would spend all of their time practising spelling, grammar, punctuation and sentence structure without ever doing any creative writing. This is a very powerful analogy: we've spent most of the time teaching children how to add, subtract, multiply, calculate and evaluate but given them little opportunity to use these in a creative way.

[3] at *canberramaths.org.au* under *Resources*

[4] Projection of the screen of a computer using the emulator onto a whiteboard provides an 'animated' whiteboard, on which graphs can be annotated, etc using a whiteboard marker.

[5] *Working Mathematically: What Does It Look Like in the Classroom?* in *Mathematics: Shaping Australia*, Proceedings of the Eighteenth Biennial Conference of The Australian Association of Mathematics Teachers, 2001; available online.

> Problem-solving, which is the creative 'goal' of mathematics, has too often been used as something 'added on' to the Mathematics lesson;[6] problems are given to the academically able students who finish their work early, or they're given to children to do for homework at the end of an exercise. Rarely are they the focus of the Mathematics lesson.

I might add that using the tools to solve problems generally and do modelling in particular is the payback or reward for students, who have spent many years learning all the tools. Too often, the reason given for learning the tools is that you will need them in Mathematics next year, a recursive reason. A better reason is that you will now use them to solve all sorts of interesting, fun and even useful problems that you could not have solved without the tools.

A mathematical model usually requires some input data, a hypothesis to explain the data and some calculations using the model to test the hypothesis or to use the model for prediction. Graphics calculators are valuable in the calculation stage if it requires lengthy or a number of similar calculations; doing these by hand limits what can be done with the model (Item 6 below). However, graphics calculators also have multiple other uses in the modelling process.

1. **As a tool in an investigation:** storing data; as a stopwatch; measuring reaction times; use with a data logger. Requires simple programs but these are readily available and easy to use.

2. **Simulation:** tossing coins, dice, etc (fair or biased); games of chance such as roulette; gambling schemes; random walks; even shuffling cards (e.g. the ProbSim app,[7] various programs [8]).

3. **Generating a table or graph of results for interpretation or further analysis.** See, for example, *The Best Shape for a Tin Can*, *Alien Attack*, *Probably Finding* π and *Statistics from Birthdays*; details in Section 3.3.2 of this volume.

4. **Data fitting and interpretation.** Analyse your data with a range of statistical tools, including fitting the data with functions (Chapter 5 here); population models (Chapter 6 here); data from the data logger.

5. **Generating statistics for interpretation or further analysis:** e.g. *Reaction Times and Statistics* (Section 3.3.2 here).

6. **Calculations using a model:** various tools for more complicated calculations. Modelling often requires the changing of parameters to see what happens to the system being modelled. Doing this by hand, even with a scientific calculator, is tedious. Once you have changed a parameter, you want to see its effect, and a graph or table is often the best way to do this; programs simplify this even further, allowing you to concentrate on the interpretation (*Population Modelling 2: Non-Exponential Models* in Volume 2; *Population Modelling 3: Matrix Models* in Volume 3).

[6] This is reinforced by such problems (often just one) only appearing at the end of the chapter in standard high-school Mathematics texts.

[7] available from *education.ti.com* under *Downloads*

[8] see *Probability and Statistics 2* in Volume 3 of this book

1.5 Why not just use computers?

Access: Any tool has to be used frequently to be useful. Graphics calculators can be used by students in all classes and at home, in fact anywhere; they are very portable. Graphics calculators can also be used in tests and exams, so the students see directly the value in using them, and they can then be examined on a much wider range of topics.

User-friendliness: Graphics calculators (some of them anyway) have been designed by teachers for ease of use, and are generally much more user-friendly than computer software. Of course, any device as powerful as a graphics calculator will take a bit of time to master. But not as long as you might think.

Conclusion: Once you start with graphics calculators, you won't want to stop.

2 Getting Started

2.1 Resetting the calculator

The following notes assume that all the default options are set. If the calculator has been used by someone else, it is a good idea to reset the calculator before proceeding.

Turn on your calculator by pressing [on], bottom left.

Press [mem] ([2nd] [+]). Press [7] for Reset. . . , [1] for RAM and [2] to Reset.

2.2 The basics

Turn on your calculator by pressing [on], the bottom left-hand key. A flashing black box ■ should appear in the upper left corner of the display screen. This screen is called the *Home screen*, and is where you type in calculations and commands. Other screens are the *Graphics screen*, the *Table screen*, the *Function-Entry screen*, the *Window screen* and the *Program-Editing screen*.

> If the black box does not appear you may have to adjust the lighting of the screen. To do this, press and release the [2nd] key, then press the up-arrow key [△]. The display is made darker by [2nd] [△] and lighter by [2nd] [▽]. Each such key combination changes the screen intensity one "step"; repeat this process (or hold down the arrow key) until the screen is properly lit.

Notice that most of the keys have yellow (84) or blue (CE) words or symbols above them. To access these yellow (blue) functions press the [2nd] key, then press the desired key for your yellow (blue) operation. Do not hold the [2nd] key down; it does not act like a shift key. Once you press [2nd], the cursor changes to a [↑]. To undo this (e.g. if you change your mind), just press [2nd] again.

To access the green letters and characters, you first press the green [alpha] key; the cursor switches to [A]. The press a letter key. For alpha lock, press [2nd] [alpha].

For example, to type the word MATHS: [2nd] [alpha] [M] [A] [T] [H] [S]. Another press of the [alpha] key returns the cursor to normal.

To clear your screen, press [clear]. If you are in the process of entering a line, one press of [clear] clears the current line. If the current line is clear and the cursor is at the beginning of a blank line, [clear] clears the whole screen. [del] deletes the character the cursor is on. [ins] ([2nd] [del]) allows you to insert characters. Move the cursor to turn it off.

From now on, we will not necessarily mention [2nd] or [alpha]. We will assume that you know to use one of these keys if the key we refer to appears in yellow (blue) or green above some other key.

2.2.1 Syntax

Calculations are performed by constructing an expression in conventional algebraic syntax (including implied multiplication), then pressing [enter] (which acts as the [=] key). Brackets are used where necessary.

Some examples of acceptable syntax are given below: each is similar to the way the expressions are conventionally written. Try them on your calculator, observing both the screen display and the final result. It is not necessary to press [clear] before each new calculation. Don't forget the [enter].

[2] [(] [5] [x^2] [−] [1] [)] [enter]	for $2(5^2-1)=48$ note implied multiplication
[4] [x^{-1}] [enter]	for $4^{-1}=0.25$
[(] [3] [+] [4] [)] [÷] [(] [5] [+] [6] [)] [enter]	for $\dfrac{(3+4)}{(5+6)}=0.6\dot{3}\dot{6}$
[math] [1] [enter]	for $\frac{7}{11}$: decimal → fraction
[3] [∧] [5] [enter]	for $3^5=243$
[(−)] [2] [−] [(−)] [3] [enter]	for $(-2)-(-3)=1$ which −? see box below
[√] [3] [)] [enter]	for $\sqrt{(3)}=1.732050808\ldots$ [√] is [2nd] [x^2]
[∛] [2] [7] [)] [enter]	for $\sqrt[3]{(27)}=3$ [∛(] is [math] [4]
-1^2 and $(-1)^2$	Which one gives the correct answer?

```
2(5²-1)
                                  48
4⁻¹
                                0.25
(3+4)/(5+6)
                        0.6363636364
Ans▸Frac
                                7/11
```

```
3^5
                                 243
√(3)
                         1.732050808
-2--3
                                   1
-1²
                                  -1
```

Note: You can omit the final bracket in a calculation.

> The calculator makes a distinction between *negative* and *minus*. When you enter 'negative 2', use the grey negative key [(−)] beside [enter], not the dark-blue minus key [−] above [+].

2.2.2 Successive commands

You can construct lengthy commands on the screen if you want before pressing [enter], but you can also do chain calculations. The result of the most recent calculation is stored in *Ans* and can be recalled using [2nd] [(−)]. Try the following key sequences. Watch where *Ans* is automatically recalled.

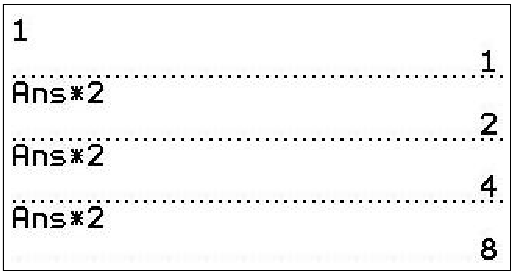
[1] [3] [+] [1] [4] [+] [1] [5] [enter]
[÷] [7] [enter]
[x^{-1}] [enter]
[√] [Ans] [)] [enter]   key in *Ans* here

```
13+14+15
                                  42
Ans/7
                                   6
Ans⁻¹
                        0.1666666667
√(Ans)
                        0.4082482905
```

What is the affect of the following?

[1] [enter]

[×] [2] [enter] [enter] [enter] . . . . . .



If you haven't typed in a new entry, pressing [enter] executes the previous entry.

2.2.3 Storing and using numbers in variables (memories)

Memories are named alphabetically, as if variables are being given values. To store a specific value in a variable (or memory), first type the value onto your screen, then press [sto], type a variable name ([alpha] followed by a single letter) and press [enter]. The value stored in the variable will not change unless you store something else in that variable name.

Let's store 3 in variable/memory P and 4 in variable/memory Q.

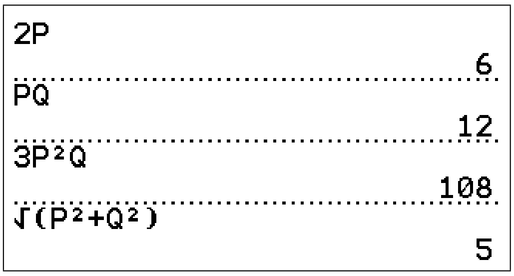
[3] [sto] [alpha] [P] [enter]   [4] [sto] [alpha] [Q] [enter]

Try the following (don't forget the [alpha]).

[P] [enter]   [2] [P] [enter]

[P] [Q] [enter]   [3] [P] [x^2] [Q] [enter]

[$\sqrt{}$] [P] [x^2] [+] [Q] [x^2] [)] [enter]



2.2.4 Recycling expressions

Next, let's try evaluating $\sqrt{P^2+3Q^2}$. Instead of typing it all over again, press [entry] ([2nd] [enter]).* $\sqrt{}(P^2+Q^2)$ should reappear on the screen. Use the arrow keys and the insert key [ins] (on the [del] key) to change the expression and press [enter].

> When you press [ins], the cursor changes from a flashing box to a flashing underline. That is the visual signal that you are in "insert mode". To get out of insert mode, move the cursor.

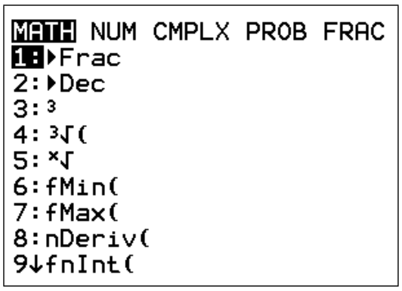
[entry] returns to the screen whatever was entered last; this expression may be edited before re-evaluating it. If you press [entry] again, you will go back to the expression before last, and so on. This is a powerful time-saver, especially for complicated expressions that you want to use again, with or without editing. Furthermore, the calculator keeps only *one* previous answer, but will remember as many previous entries as it can, up to 128 characters on an 84.

*On the CE and more recent versions of the 84 operating system,[9] you can also scroll up through previous calculations using the up arrow. Press [enter] to paste the highlighted entry into the command line for editing.

[9] You can download the latest operating system from the TI website *education.ti.com* and install it on your calculator using TI-Connect/TI-ConnectCE (also available at *education.ti.com*).

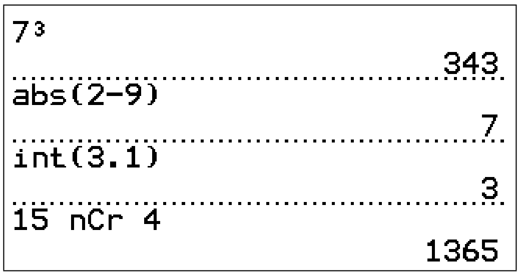
2.2.5 Using menus

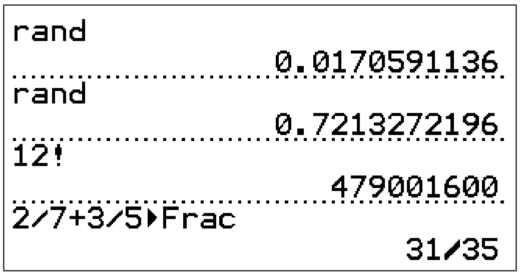
The calculator is menu-driven, which explains why the keyboard is relatively uncluttered. For example, many functions are located in the [math] menu. Here you will see the four (five) sub-menus displayed, with headings MATH, NUM, CPX (CMPLX) and PRB (PROB) (plus FRAC on a CE). The arrow keys allow you to move between these.



To access a menu function, press its number (or use the arrow keys and [enter]). Generally you construct an expression just as you would on paper. To leave a menu without selecting a function, press [clear], which returns you to the Home screen. Try the following examples.

[7] [math] [3] [enter]	for $7^3 = 343$
[math] [▶] [1] [2] [−] [9] [enter]	for $\lvert 2-9 \rvert = 7$: absolute-value function
[math] [▶] [5] [3] [.] [1] [)] [enter]	for $[3.1 = 3$, greatest-integer function
[1] [5] [math] [◀] [◀] [3] [4] [enter]	for $^{15}C_4 = 1365$
[math] [◀] [◀] [1] [enter] [enter]	successive random numbers on $(0, 1)$
[1] [2] [math] [◀] [◀] [4] [enter]	for $12! = 479,001,600$
[2] [÷] [7] [+] [3] [÷] [5] [math] [1] [enter]	for $2/7 + 3/5 = 31/35$ as a fraction





2.2.6 Defining and evaluating functions

The top row of keys is for defining, graphing and generating tables of functions.

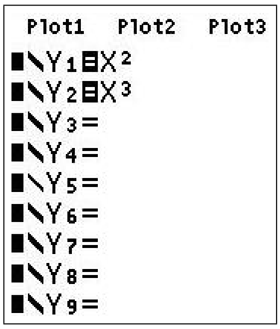
The calculator has two display modes, set in the [mode] menu (scroll down on an 84).

MATHPRINT tries to display maths as it would appear in proper maths expressions, such as in textbooks; nice for output, but can complicate input sometimes.

CLASSIC has inputs and outputs on one line. This is easier for inputs, and is the mode used here. Select it from the [mode] menu using the arrow keys and [enter].

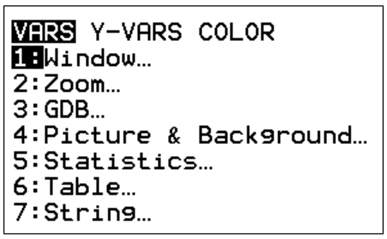
Defining a function is a simple task. For example, to enter the function $f(x)=x^2$, press [y=] [X,T,θ,n] [x^2]. The [y=] key brings up the Function-Entry screen with 'empty' functions Y_1 to Y_7 or Y_9, depending on the model.[10] The [X,T,θ,n] key provides the appropriate independent variable (X in this case) and the [x^2] key squares whatever precedes it. Note that, when you define a function, the = sign after its name is highlighted. This means the function is 'selected' for tables and for graphing. We will take up graphing soon.

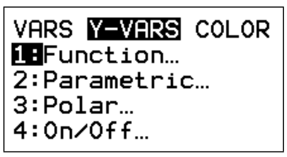
Next enter $g(x)=x^3$ in Y_2: in the Function-Entry screen, press [enter] or [▼] to move to the next line. Then press [X,T,θ,n] [math] [3] or [X,T,θ,n] [∧] [3]. Return to the Home screen by pressing [quit] ([2nd] [mode]).

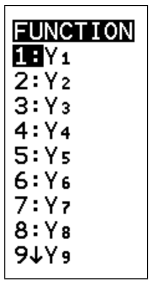


Now that we have defined some functions, we need to know how to use them. For example, how do we find out what $g(2)$ equals? g is known to the calculator as Y_2, so we have to refer to it by that name.

Press [vars] and select Y-VARS with the right arrow. That brings up a menu of possible types of function names. So far, we are interested only in the first type, called *Function*, so press [1].[11] Now you see a list of the function names: press [2] to put Y_2 on the Home screen.



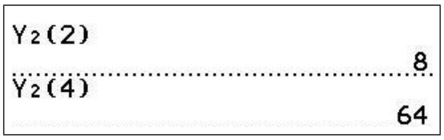




[10] There are actually 10 (scroll down), but only 7 or 9 (CE) fit on the screen.

[11] [alpha] [trace] (F4) gets you here too on the CE and later versions of the 84 operating system.

To evaluate Y2 (2), just finish the expression with (2) , then enter . The calculator should display 8.



Now calculate $g(4)$. You can either repeat the steps we just did — or you can think about an easier way to do this (Section 2.2.4).

> You can have as many as 10 functions defined at one time, but you may not want to graph them all at once. Turn off (deselect) any or all of them in the y= screen by moving the cursor over the = sign and pressing enter . Turn them back on the same way.

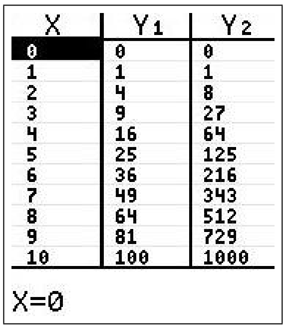
You can choose the type of line for each function graphed (and its colour on a CE) by moving the cursor to the left of its = sign and pressing enter .

2.2.7 Using a table

A quicker way to find a number of function values is to use the table feature. Press y= to check that you still have both Y1 and Y2 selected (selection for graphing also means selection for table-building).

Then press tblset (on window) to get the TABLE SETUP screen (the table 'window'). Set *TblStart* (the starting value of X) to 0 and Δ*Tbl* (step size between values of X) to 1. *Auto* should also be selected for both independent and dependent variables (use the cursor and enter if not).

Now press table (on graph) and see what happens.



X	Y1	Y2
0	0	0
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	343
8	64	512
9	81	729
10	100	1000

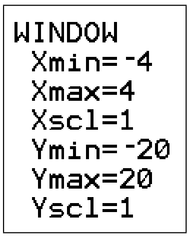
X=0

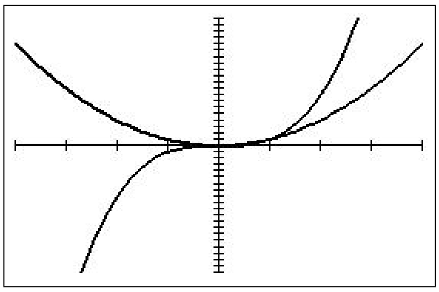
Try scrolling the cursor up and down the X column, and watch the changes in the Y1 and Y2 columns. In particular, go beyond the bottom number in the downward direction and beyond the top number in the upward direction. Move the cursor into the Y columns, and watch what happens at the bottom of the screen. Move the cursor *above* the table in either of the Y columns, and again watch the bottom of the screen. If we had more functions defined and selected, they would all get tabulated, and we could find their values by scrolling to the right if they didn't all fit on the screen.

Finally, we can change the spacing of X values in the tables by changing Δ*Tbl* in tblset . Try it! Use quit to return to the Home screen.

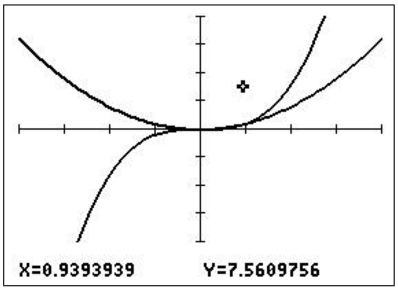
2.2.8 Graphing functions

To graph the formula-defined functions that are selected in the y= screen, just press the graph key. The default window for graphing (hopefully set) is from −10 (Xmin, Ymin) to 10 (Xmax, Ymax) in both the horizontal and vertical directions. You can change that by pressing the window key. Make the window go from −4 to 4 in the X direction and from −20 to 20 in the Y direction (which − sign?). Press graph again to see the same functions graphed in the new window.





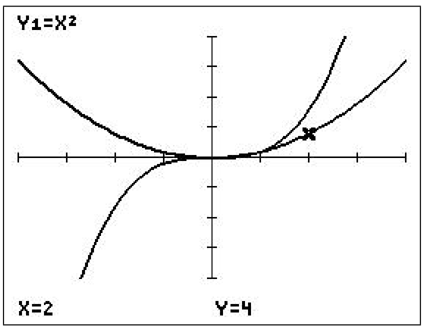
You may find that your Y axis now looks a little funny. Go back to the window screen. The entries for Xscl (X scale) and Yscl are the distances between tick marks on the axes. With both set at 1, these marks appear at every integer value in both directions. With the Y range now extending 40 units, that's a lot of tick marks. Change Yscl to 5 and graph again. Better?



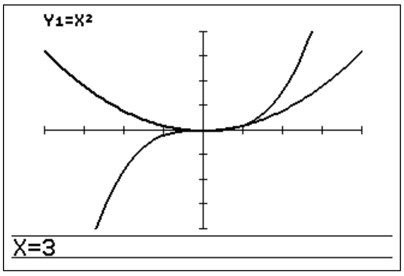
Move the cursor around the screen with the left- and right-arrow keys. The cursor coordinates are shown at the bottom of the screen.

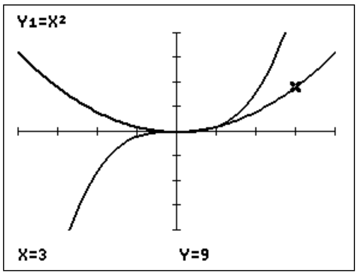
2.2.9 Tracing and zooming

Let's try some other keys. With your graph on the screen, press trace. A flashing cursor should appear on the first graph half way between the left side and the right side of the screen. (It happens that this point is on both graphs in this case.) Notice that the coordinates of this point on the graph appear at the bottom of the screen and the formula for Y_1 in the upper left corner. Use the right and left arrows to move along the graph of Y_1. Press the up and down arrows.

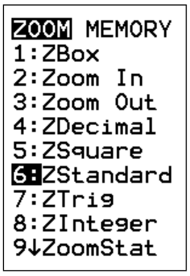


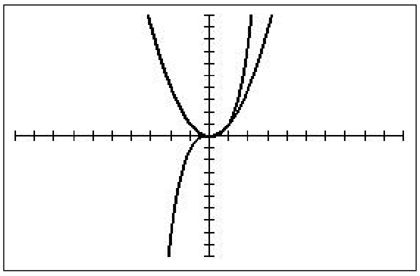
To go to a specific X value on the graph (for example to find out the corresponding Y value), just type in the X value (3 here) and press enter. If the value lies in the current window, the cursor will move there.

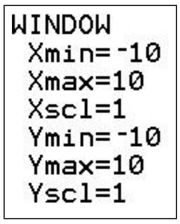




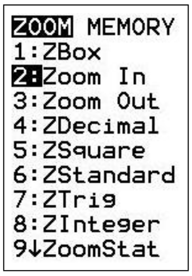
Now try zoom 6 for *Zoom Standard*. This is an easy way to recover the standard window with Xmin = −10, Xmax = 10, Ymin = −10, and Ymax = 10. As we started in the standard window in Section 2.2.8, the graphs are redrawn as we first saw them.

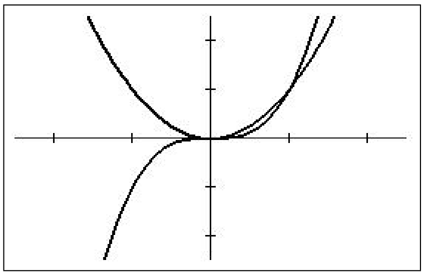


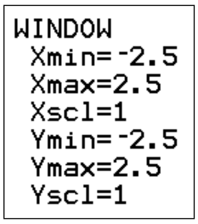




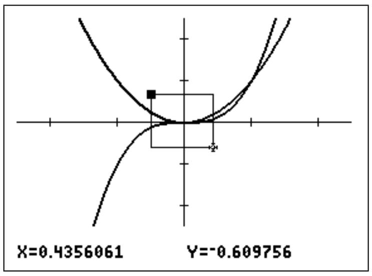
Next try zoom 2 to *Zoom In*. A small cursor appears in the middle of the screen, although it may be obsured by the axes. Move it around with the arrows until you can see it and then press enter. The calculator now zooms in (magnifies), centred on the spot where you left the small cursor (the origin in the figure below). Press window to see the effect of *Zoom In*.

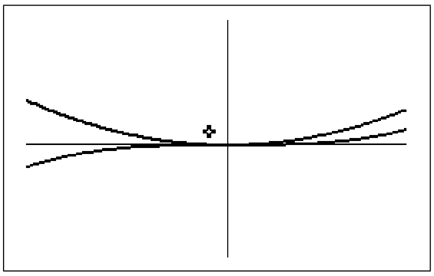


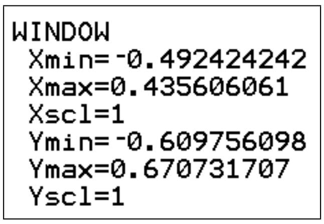




Press zoom to go back to the Zoom menu. Try *ZBox*; again a small cursor appears on your screen. Move it a bit up and a bit left and press enter. The cursor now has a box shape. Move the active cursor down and to the right; the screen displays an outlined box with a fixed mark at one corner and the active cursor at the opposite corner. Move the active cursor to enclose some interesting part of the graphs, and press enter again. The graphs are redrawn with your selected box taking up the whole screen. Press window to see the effect of *ZBox*.







There are several more types of zooms; read the manual to see what these do. You might like to look at ZOOM MEMORY too.

2.2.10 Finishing up

If you don't press any keys for five minutes or so, the calculator turns itself off. However, you can also turn it off with (surprise!) [2nd] [on]. All settings, function definitions, variable values, etc are remembered when the calculator is off. If you turned it off yourself, the next time you turn it on, you will see the Home screen, exactly as you left it. If it turned itself off, the next time you turn it on, you will see the screen you were on when it turned off.

2.3 Significant digits and calculations

When we write down a numeric answer to a problem, we have to make some decision as to how many digits we put in the answer. Firstly we have to decide how accurate our answer is — this will depend on the accuracy of the data we use in our calculations and on whether we introduce any further loss of accuracy by our method of calculation. These considerations are discussed below. Secondly, we have to decide what is a *sensible* number of digits to put in our answer. We wouldn't give the distance to a star to a number of digits that takes us down to millimeters, even if we knew it that accurately.

In specifying the accuracy of an answer, we usually give the number of decimal places (DP) or the number of significant digits (SD) that we think are appropriate. We will use significant digits in this course, because it fits in well with scientific notation for specifying numbers, for example 3.56×10^{-5}.

Examples

1. 0.0036 has 2 SD — leading zeros are not significant. We could also write this number as 3.6×10^{-3}, making the number of significant digits clear.

2. 2 may have 1 SD or may also be an exact number and so implicitly have an infinite number of SD.

3. 2. has 1 SD.

4. 2.00 has 3 SD — trailing zeros after the decimal point are significant.

5. $240,000,000$ has 2 SD — trailing zeros before a decimal point are not significant unless specified as being so. Write as 2.4×10^{8}.

6. Be careful with rounding. If your answer is an approximation, you should specify the number of significant digits which are accurate.

$$\begin{aligned}\pi &\approx 3.141592654 \quad (10\,\text{SD})\\ &\approx 3.14 \quad (3\,\text{SD: rounding down})\\ &\approx 3.142 \quad (4\,\text{SD: rounding up})\\ &\approx 3.1416 \quad (5\,\text{SD: rounding up})\end{aligned}$$

There are at least three sources of concern about significance of digits in an answer.

- **The accuracy of available data**

  You should not expect more significant digits in any answer than there are in the least accurate input to the calculation. *Calculation steps never add significant digits, though your calculator will happily add digits!*

  Example: A population of $240,000,000 = 2.4 \times 10^8$ grows by 2% per year for 3 years. What's the population after 3 years?

  Numerically, the answer is $2.4 \times 10^8 \times 1.02^3 = 254,689,920$. However, the original number had only 2 SD (2 and 4). The best answer to the question is "about $2.5 \times 10^8 = 250,000,000$". The answer $254,689,920$ is definitely wrong.

- **The finite precision of your calculating device**

  Subtraction of nearly equal numbers can be a real significance killer

  Example: If we subtract two numbers that only differ in the tenth digit, the answer has only one significant digit.

  Adding or subtracting numbers that are very different in magnitude can also lead to inaccuracy

  Exercise: We all know that $\mathrm{A} + \mathrm{B} - \mathrm{A} = \mathrm{B}$.

  Store 10^6 in memory A.[12]

  Store $\sqrt{(}2) \times 10^{-5}$ (correct − sign?) in B.

  Evaluate $\mathrm{A} + \mathrm{B} - \mathrm{A}$.

  Repeat[13] with $\mathrm{A} = 10^7$, 10^8, 10^9. Explain.

- **Loss of significance due to the way we manipulate numbers**

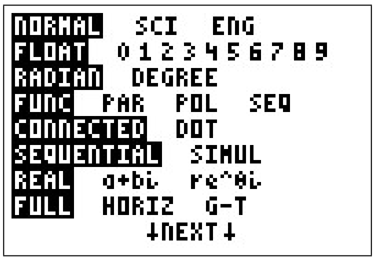
  Don't discard digits in an intermediate result. The only time you should round off is at the end of a calculation. Preferably use your calculator to do the calculation all in one go — it keeps 14 digits.

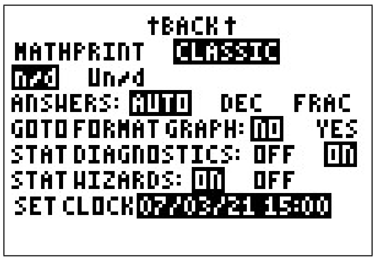
[12] [EE] [6] [sto] [alpha] [A] [enter]
[13] Remember the [entry] key.

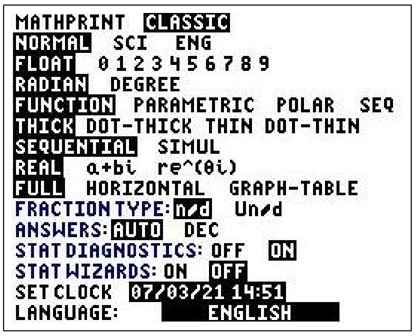
2.4 The mode and format menus

2.4.1 Mode

First we'll explore the modes of the calculator: press mode. Your screen fills with words and numbers. This is actually a lot of different menus, one per line. The Mode menu takes up two screens (scroll down) on the TI-84 (left-hand figures below), only one on the CE (right-hand figure below).







The line numbering here is from the CE.

Line 1

MATHPRINT/CLASSIC. See page 9. This line is on the second screen of the 84.

Line 2

This determines how the calculator displays numbers. **NORMAL** displays numbers in the usual fashion until they become too big or too small; it then changes to scientific notation. **SCI** gives scientific notation, that is all numbers in the form $\pm a.bcd \cdots \times 10^q$. Try calculating $22 \div 100$. You should see 2.2E^{-}1, the scientific notation for 0.22. In most calculator and computer systems, E (exponent) in a number stands for "times 10 to the power". **ENG** gives scientific notation but with the power of 10 a multiple of 3, with the preceding number adjusted appropriately.

Return to mode. Switch back to **NORMAL**.

Line 3

This line controls the number of decimals displayed. **FLOAT** means no fixed number (the calculator chooses appropriately), while the rest of the line allows you to set the number of decimal places displayed. The calculator always works with 14 digits, irrespective of the setting here.

Line 4

In the third line, you can choose between **RADIAN** and **DEGREE** angle measurement. Be careful! *The wrong setting here is the most common cause of error in graphing and analysing trig functions.* **RADIAN** preferred for Maths.

Line 5

This line indicates the different types of graphs that are possible. We used **FUNCTION** for 'function graphs', $y = f(x)$; this is the graph mode we will use most of the time. The other modes are **PARAMETRIC** ($x = f(t)$, $y = g(t)$), **POLAR** ($r = f(\theta)$) and **SEQUENCE** ($u(n)$, $v(n)$, $w(n)$).

Line 6

This controls whether the plotted 'dots' in a graph are Connected by lines or not. On the CE, you can have both thick and thin dots and lines. You can set these for individual functions in [y=].

Line 7

SEQUENTIAL means the functions defined are graphed sequentially; **SIMUL**(TANEOUS) means they are graphed simultaneously.

Line 8

Choose **REAL** to work with real numbers only. In working with complex numbers, numbers can be displayed in either Cartesian ($a+bi$) or polar ($re^{\theta i}$) form. See *Complex Numbers* in Volume 3 of this book for more details.

Line 9

This line determines how the screen is divided up. **FULL** is full screen for whatever you are displaying. **GRAPH-TABLE** displays the graph and table of a function on the one screen. In **HORIZONTAL**, the top half of the screen shows the graph, the bottom half of the screen the Home screen, Table, y= or Window. Full-screen menus (e.g. mode) will temporarily use the whole screen, then restore the split screen (if it is still selected) when you exit.

Line 10

How fractions are displayed, proper or improper. Take your pick.

Line 11

How answers are displayed. Leave on AUTO.

[*Line 12 on a TI-84 just sends you to the Format screen if you select YES.*]

Line 12

STAT DIAGNOSTICS displays the various parameters associated with fitting a curve to points when ON. Recommend ON.

Line 14

STAT WIZARDS helps set up a statistics plot rather than you doing it all in [stat plot]. Useful.

Line 15

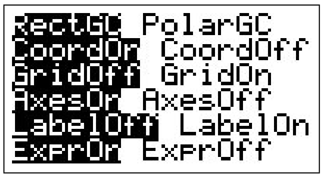
This line allows you to see and set the clock.

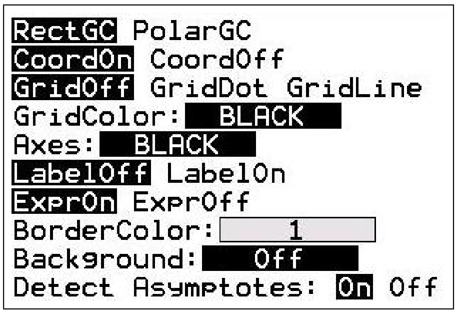
Summary of Mode

The default mode settings (the left-hand choices of Lines 2–9) are just right for most of what we will be doing. If you have any other choices selected, return them all to default settings now. At other times, you may want to return to the Mode menu to change the form of numerical display, angle measurement or graphing options.

2.4.2 Format

Select format (on the zoom key). The Format screen works just like mode. The CE has a few more options here because it offers colour and two grid types.





Line 1

In trace, the cursor coordinates on the Graphics screen can be displayed in **RectGC** (x, y) or **PolarGC** (r, θ) coordinates.

Line 2

In Sections 2.2.8 and 2.2.9, you used the arrows to move around the Graphics screen, with the cursor coordinates appearing at the bottom of the screen. If you choose **CoordOff**, the cursor will appear and move, but no coordinates will appear.

Line 3

GridDot and **GridLine** tells the calculator to use the tick marks you set on the axes (with Xscl and Yscl) to create either a grid of dots or a grid of lines on the Graphics screen of the CE. The 84 has only a dot grid. **GridOff** (the default) turns this option off.

Line 4 on a CE allows you to choose the colour of the grid.

Line 5

The axes on an 84 graph can be either on or off. On the CE, they can be off, or on and different colours, including black.

Line 6

This turns on or off the display of axis labels, i.e. the characters x and y. However, the placement of these characters on the screen is not useful, because they appear in strange locations (try it and see); most of us can figure out which is the X axis and which is the Y axis anyway. Leave it set on the default setting **LabelOff**.

Line 7

This line allows you to turn on/off the equation of the function in the top left corner of the Graphics screen when you are using trace. Sometimes this expression obscures part of the graph on an 84, and it is useful to be able to turn it off.

The CE has another three lines.

Line 8

BorderColor: the Graphics screen does not occupy the whole screen; it has quite a wide border around it for which you can choose three different colours and white.

Line 9

Background: you can use one of the Images (see the manual) as a screen background, rather than just a plain screen.

Line 10

Detect Asymptotes: on a graph (just a series of points calculated from the function) in Connected mode (Line 6 in Mode), the calculator tries to join successive points to form a continuous curve. If the function has a vertical asymptote, it will therefore draw an unwanted almost vertical line at the asymptote. With **Detect Asymptotes On**, it won't do this.

2.5 The graphics screen and accuracy

2.5.1 The graphics screen

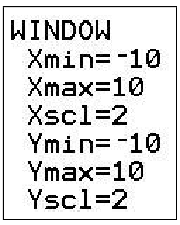
Modified from D. Pence, *Calculus Activities for TI Graphic Calculators*, 2nd ed, PWS Publishing, 1994.

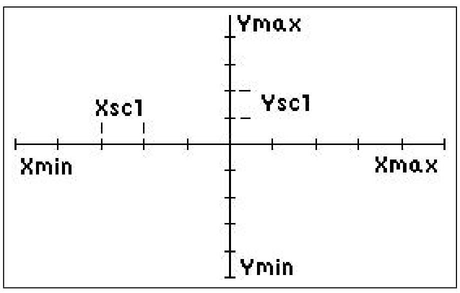
Before graphing a function f on a graphics calculator, you must first specify the range of X values to be considered (i.e. restrict the domain to a finitely bounded interval of numbers) and the range of Y values to be allowed. Setting these ranges defines the scales and locates the coordinate axes on the screen. Generally we will use the estimated range of f as the Y range. However, the true range of f may be an unbounded interval, and so it will not be possible to set the entire Y range.

The calculator screen consists of rows and columns of little rectangles, called *pixels*. Compared to a television screen or computer monitor, the pixels on the TI-84 screen are relatively large; the screen consists of a grid 96 pixels across and 64 pixels high.[14] You can see the individual pixels by turning up the contrast. To do this, press and release 2nd, then hold down the up-arrow key. *You will notice numbers in the top right-hand corner which indicate the level of contrast. The greatest contrast is 9. If you need a contrast level of 8 or 9 for normal use, you should think about some new batteries.* Turn the contrast back down again by pressing and releasing 2nd, then holding down the down-arrow key until you reach the correct setting for you.

On a TI-84CE, the screen consists of 164 rows and 264 columns of pixels, so individual pixels are not visible. You adjust the contrast/brightness the same way as on the 84.

Any point (X, Y) on the screen lies in the range $\text{Xmin} \leqslant \text{X} \leqslant \text{Xmax}$, $\text{Ymin} \leqslant \text{Y} \leqslant \text{Ymax}$, where Xmin, Xmax, Ymin and Ymax are set using window.





[14]The screen used for graphics is actually 95×63 pixels.

The first *column* of pixels on the left has the X coordinate *Xmin*, and the last but one *column* of pixels on the right the X coordinate *Xmax*. Similarly, the *rows* of pixels have equally spaced Y values between *Ymin* and *Ymax*. There is a similar grid of pixels for any computer-drawn coordinate graph; on the CE, the pixels are so small that the eye cannot pick them out.

Each pixel has coordinates (the coordinates of the centre of the pixel) that appear at the bottom of the screen as you move the cursor around with the arrow keys. However, each pixel actually represents a region in the plane (i.e. infinitely many points). If a point to be plotted lies somewhere in the region represented by a pixel, the pixel is turned on (darkened). In Trace mode, the X value at the bottom of the screen is the X coordinate of the pixel, but the Y value is the value of the function at that X value. This is usually different to the Y coordinate of the pixel, but must lie in the range of Y values covered by that pixel.

The X axis will appear on the screen if Ymin $\leqslant 0 \leqslant$ Ymax. Scale marks are drawn along the axis at a regular spacing *Xscl*, also set in [window]. Similarly, the Y axis will appear if Xmin $\leqslant 0 \leqslant$ Xmax, with scale marks spaced by *Yscl*. No scale marks appear if Xscl/Yscl is set to 0. If Xscl or Yscl is too small, the scale marks will run together and the axis will look as though there is a parallel line running right next to it. This is often confused with the graph of the function you are trying to plot.

There are several commands in the [zoom] menu which provide convenient ways to change the window settings. These commands also contain an implied graph command.

[zoom] [6] (ZStandard) gives window settings $[-10, 10, 1] \times [-10, 10, 1]$.

[zoom] [7] (ZTrig) gives window settings of approximately $\left[-2\pi, 2\pi, \frac{\pi}{2}\right] \times [-4, 4, 1]$ on an 84, $\left[-2.75\pi, 2.75\pi, \frac{\pi}{2}\right] \times [-4, 4, 1]$ on a CE, suitable for most trigonometric functions.

[zoom] [5] (ZSquare) makes the scales on the X and Y axes the same. For equal scales on an 84, (Xmax−Xmin) is $95/63 \approx 1.5\times$(Ymax−Ymin). ZSquare is useful if the shape of the function you are plotting is important, such as a semi-circle.

It is often useful to have a setting in which the coordinates of adjacent columns and rows differ by 0.1, i.e. the pixel widths and heights are both 0.1 . This is achieved by using [zoom] [4] (ZDecimal).

Note: On a CE in Trace mode, the cursor moves in steps of two pixels when you press the right-arrow or left-arrow keys. For example, in ZDecimal, the free cursor moves in steps of 0.05 (the width of a pixel), the Trace cursor in steps of 0.1.

2.5.2 Function graphers — Getting the picture

Modified from T.P. Dick and C.M. Patton, *Student Guide to Using Technology in Calculus*, PWS-Kent, Boston, 1992.

A graph provides a powerful interpretational tool by giving us a visual picture of the input-output pairs a function process produces, but it requires much time and effort to prepare graphs by hand. With the availability of computer and calculator graphics technology, we have a much greater opportunity to exploit graphical representations of functions.

Be forewarned: the graphical evidence provided by a machine can be open to perceptual illusions and therefore to misinterpretations. To make intelligent use of graphical tools, it is important to understand their limitations. In other words, getting the most out of graphics technology requires not only knowing how it can be used, but also how it *can't* be used. Let's look at some of the issues you must be concerned with when using graphing technology.

First of all we need to understand how a graph is produced and displayed by a machine. The screen of a calculator or computer is divided into a rectangular grid of small square picture elements called *pixels*. Each pixel has coordinates corresponding to a single point in the plane, but a pixel does not really represent a point. Rather, a pixel represents a small rectangle containing infinitely many points. The specific point given by the coordinates of the pixel may represent the centre or a corner of the pixel, depending on the particular machine or software (*on the TI-84, it's the centre*). If we want the machine to indicate a certain point in the plane, we have to light up or darken the particular pixel containing that point.

The viewing window

Every graphics package on a computer or calculator has a necessarily limited screen. You might think of this screen as a window from which you can view part of the Cartesian plane. By moving this window around the plane, we can focus our attention on various parts of the graph of a function. This window is also a *lens* through which we can obtain both close-up and distant views of the graph by changing scale. Finding the best window locations and scales are navigational skills for finding our way about a function's graph.

Graphical behaviour can be hidden

- by lying beyond the bounds of the viewing window,
- by scale — zooming in obscures global information about the graph; zooming out obscures local information or detail about the graph,
- by numerical limitations — the choice of which pixels to light up or darken is determined by numerical computations, which in turn are subject to the usual round-off, cancellation, underflow and overflow errors that may occur.

Exercise: Graph the function defined by the formula

$$f(x) = \frac{(x^3 - 1)}{(x - 1)},$$

using Y1 and window parameters $[-6, 5, 2] \times [-2, 10, 2]$. Be careful with brackets.

(a) What is $f(1)$? Check your calculator's reaction to this calculation by evaluating Y1(1) (page 10).

(b) Does this problem show up on your graph? Use [trace] to investigate. Explain.

(c) Now change the X window to $[-4.7, 4.7, 2]$ and repeat (b). Why the difference?

(d) Move the cursor using [trace] close to, but not at, X = 1 and zoom in using [zoom] [2] (page 12).

Do this repeatedly — [trace] close to X = 1, then [zoom] [2], about 10 times — until an irregularity appears.

Zoom in several more times. Why do you think this might happen? Look at the window.

The graph of f has a *hole* at $x = 1$, as the function is not defined there. However, a machine plot of this function's graph could have several different appearances near $x = 1$:

- the graph may appear to be continuous if $x=1$ falls in between the X coordinates of two adjacent pixels (but not close enough to either to have the function value affected noticeably by numerical imprecision);
- there may be a missing pixel if the X coordinate of a pixel is exactly 1;
- there may be a jagged jump or spike if the X coordinate of a pixel is very close, but not equal, to 1, due to numerical imprecision.

You are more likely to observe the visual effects of numerical imprecision at small scalings. Even a continuous function's machine-plotted graph may break apart under repeated zooms because the function cannot be calculated accurately enough.

Exercise: Graph the function $\left(2\cos(x)-2+x^2\right)/x^4$ (RADIAN) using a window of

(a) $[-10,10,0]\times[-0.1,0.2,0]$

(b) $[-0.001,0.001,0]\times[-0.1,0.2,0]$.

What do you observe? This is another example of numerical instability — the calculator cannot calculate $\cos(x)$ accurately enough when its argument is very close to 0.

The dimensions of the viewing window are specified by the parameters Xmin, Xmax, Ymin and Ymax. The TI-84Plus graphing screen is 95 pixels wide by 63 pixels high (the actual screen is 96×64, but only 95×63 is used for graphing). From these numbers we can calculate the dimensions of a pixel.

Example: Given a window of $[-10,10]\times[-5,5]$, find the dimensions of an TI-84Plus pixel.

On the TI-84Plus, the coordinates of a pixel correspond to the centre of the pixel. The pixels are (effectively) touching, so that each interval between adjacent pixels is equal to the width of a pixel. As there are 95 pixels across the width of the screen, there will be 94 pixel-width intervals between the centre of the leftmost pixel representing *Xmin* and the centre of the rightmost pixel (of the 95) representing *Xmax*. Therefore the *width* of an 84Plus pixel is

$$\Delta\text{X} = \frac{\text{Xmax}-\text{Xmin}}{94} = \frac{10-(-10)}{94} = \frac{20}{94} \approx 0.21.$$

Similarly, as there are 63 pixels vertically, the *height* of each pixel is

$$\Delta\text{Y} = \frac{\text{Ymax}-\text{Ymin}}{62} = \frac{5-(-5)}{62} = \frac{10}{62} \approx 0.16.$$

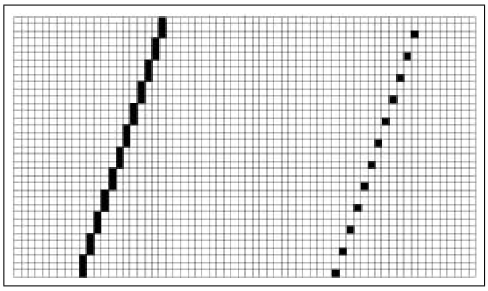
ΔX and ΔY can be accessed through the [vars] Window menu. These values are calculated whenever you plot a graph.

Exercise: If the viewing window on a TI-84Plus is $[-4.7,4.7]\times[-3.1,3.1]$, find the dimensions of each pixel. This is often a useful window to use; it is set automatically by ZDecimal in [zoom].

When graphing a function f, the calculator starts with the first (leftmost) column of pixels. It calculates the ordered pair $\left(\text{X}, f(\text{X})\right)$ using the X coordinate of that column (Xmin) as the value of X, and darkens the pixel in that column whose Y coordinate is closest to $f(\text{X})$, provided $f(\text{X})$ is within the vertical range of the window, i.e. between Ymin and Ymax. This process is then repeated for each column of pixels from left to right.

In *Dot mode*, the calculator will darken at most one pixel in each column (so the function graph on screen will pass the vertical line test). In *Connected* mode, the function grapher will darken additional pixels to give the visual perception of an unbroken graph. The figure below shows two TI-84Plus graphs of the same line; one is plotted in *Connected* mode, the other in *Dot* mode.

In either case, note that the calculator graph of a function is simply a finite collection of darkened pixels, whereas the true graph consists of infinitely many points.



3 Coordinate Geometry

3.1 Terminology and useful keys

- **Home screen:** the screen where you type in calculations and commands.

  Press [quit] ([2nd] [mode]) to return to the Home screen from any other screen.

- **[clear] :** clears the line you are currently typing. If the current line is already clear and the cursor is on a blank line, [clear] clears the whole screen. In a menu, [clear] exits the menu if you don't want to select a menu item.

- **[del] [ins]** : [del] deletes the character the cursor is on. [ins] allows you to insert characters before the character the cursor is on. To turn [ins] off, press [ins] again or just move the cursor with the arrow keys.

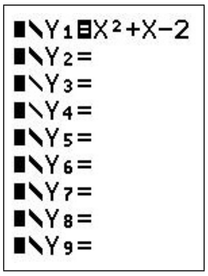
3.2 Basic operations

3.2.1 Graph $f(x) = x^2 + x - 2$ for $-3 < x < 2$

- Press **[y=]**: set $Y_1 = X^2 + X - 2$.

  The independent-variable key [X, T, θ, n] gives X. Note the highlighted = sign, which means the function will be plotted when you press [graph].

  Move the cursor over the = sign and press [enter] to toggle the function off/on.

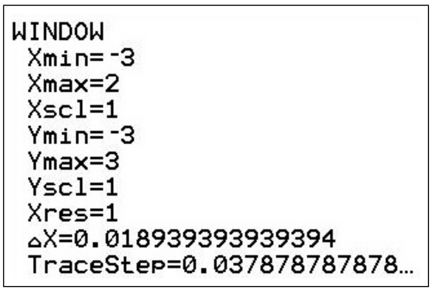


- Press **[window]**: specify the viewing window as shown in the figure.

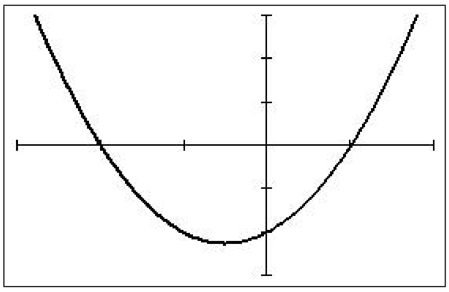
  Note the difference between the blue [−] key (subtract) and the white [(−)] key (change sign).

  Use [enter] to move between values.

  Xscl, Yscl are the distances between tick marks on the axes (0 gives no tick marks). The last 3 parameters are not relevant to general graphing.



- Press **[graph]**: graph the function.



window $[-3, 2, 1] \times [-3, 3, 1]$

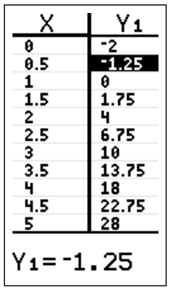
3.2.2 Generating a table of function values

- Press **y=** and make sure the = signs of the functions you want are highlighted.

- Set the table 'window' using **tblset**:
  TblStart = 0, ΔTbl = 0.5, Auto Auto;
  select with the cursor and enter.
  This generates X values automatically, starting at 0 and incrementing in steps of 0.5.

- Press **table** (on the graph key); *scroll up and down in the X column, down in the Y column.* Note the display of the highlighted number at the bottom of the screen.

  Scroll to the function header to see the formula. You can actually change the formula here by pressing enter, editing it and pressing enter again.

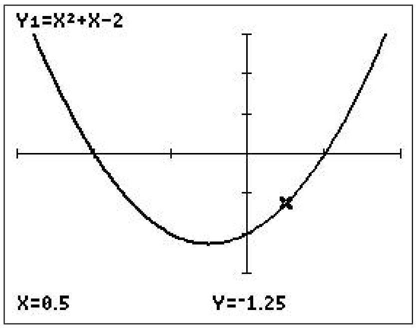


- To enter your own X values, set *Indpnt* to *Ask* in tblset and press table.

3.2.3 Find (estimate) $f(0.5)$

- **From a table** — see above.

- **On the graph**
  – Press **trace** (*top row of keys*).
  – Type in the X value, 0.5, and enter to move to the desired point on the graph. Note the coordinates at the bottom of the screen.
  – You can use the left and right arrow keys to move the cursor along the curve but note the problem that arises when trying to reach X = 0.5.
  – The up and down arrows move between functions if there is more than one graphed.



window $[-3, 2, 1] \times [-3, 3, 1]$

- **On the Home screen**

  – The calculator knows f by the name Y1. We need to evaluate Y1 (0.5).
  – Y1 is in the vars Y-VARS Function menu: vars ▶ 1 1. You can't just type Y 1.

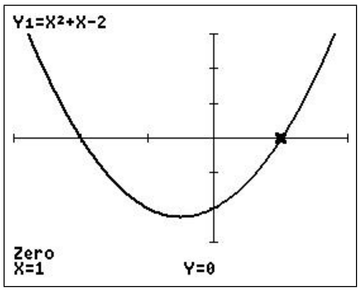
  – Follow with standard function notation (0.5) and enter.

- *Answer*: $f(0.5) = -1.25$.

3.2.4 Find the zeros of $f(x)=x^2+x-2$

- Graph the function with the previous window $[-3,2,1]\times[-3,3,1]$.
- For a rough estimate of a zero, press trace and move the cursor as close as possible to the zero. Read the cursor coordinates at the bottom of the screen. Watch the Y coordinate to see when it changes sign. Zooming in on the zero — zoom 2, move the cursor to the zero and press enter — will produce greater accuracy with this method.
- For a more accurate estimate, use *zero* in the calc menu (on trace).[15]

  The calculator asks for a left bound: move the cursor somewhere to the left of the zero you want (or type in a value) and press enter. For the right bound, move the cursor somewhere to the right of the zero you want (or type in a value) and press enter. Similarly for *Guess*. The guess doesn't have to be spot on. The calculator will then approximate the zero. Your bounds and guess will determine which zero you find if there are several.



window $[-3,2,1]\times[-3,3,1]$

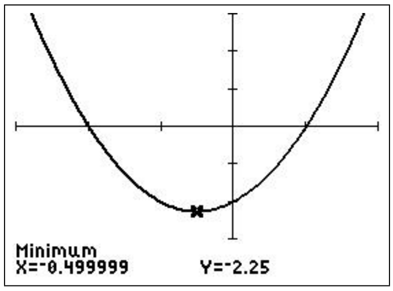
- *Answer*: $f(x)=0$ when $x=-2$, 1.

3.2.5 Find the minimum of $f(x)=x^2+x-2$

- For a rough estimate, press trace and move the cursor as close as possible to the minimum. Read the cursor coordinates at the bottom of the screen. Watch the Y coordinate. Zooming in on the zero — zoom 2, move the cursor to the zero, press enter — will produce greater accuracy with this method.

  Note: The minimum can be found exactly here using trace because X=−0.5 happens to correspond to a pixel.

- For a more accurate estimate, in general, use *minimum* in the calc menu. *minimum* works the same way as *zero* above. Use the bounds and the guess to pick out the minimum you want if there is more than one.
- Finding maxima using the *maximum* command works in exactly the same way.



window $[-3,2,1]\times[-3,3,1]$

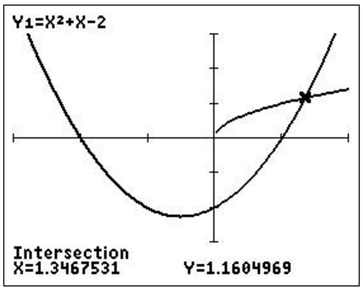
- *Answer*: the minimum value of $y=-2.25$ occurs at $x=-0.5$, rounded to 5 decimal places (the claimed accuracy).

[15]If you have more than one curve graphed, select the one you want with the up/down arrow keys before the next step.

3.2.6 Solve $x^2+x-2=\sqrt{x}$

- Graph $Y_1=X^2+X-2$ and $Y_2=\sqrt{}(X)$ using the previous window.
- For a rough estimate, press trace and move the cursor as close as possible to the intersection. Read the cursor coordinates at the bottom of the screen. Zooming in on the zero — zoom 2, move the cursor to the intersection, press enter — will produce greater accuracy with this method.
- For a more accurate estimate, use *intersect* in the calc menu.
- Press enter to select each curve.*

  Move the cursor to provide the *Guess* (or type in an X value) and press enter. Your guess will determine which solution you find if there are several.



window $[-3,2,1]\times[-3,3,1]$

  *The cursor starts on the first curve (usually Y_1). The cursor automatically moves to the second curve after you press enter the first time. If you have more than two curves graphed, select the curves you want with the up or down arrow keys (look at the top left of the screen) before pressing enter.

- *Answer*: $x^2+x-2=\sqrt{x}$ when $x=1.347$, $y=1.160$, rounded to 3 decimal places.

3.2.7 Graph styles and shading regions

There are eight (seven) possible graph styles and 15 colours on an 84CE (84Plus), selected by moving the cursor to the left of the Y in the function definition in y= and pressing enter. Try graphing a function with each style. Shading can be useful for inequalities and Linear Programming (Chapter 4).

- thin line
- thick line
- shade above the function
- shade below the function
- moving ball with trail
- moving ball without trail
- thick dotted line (not on 84Plus)
- thin dotted line

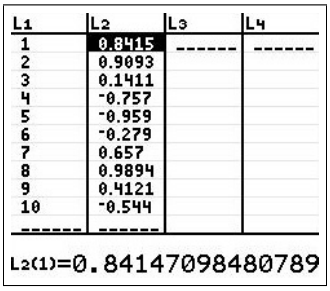
For more intricate shading, use the *Shade* command in the draw DRAW menu. See the manual for details.

3.2.8 Graphing data points and points joined by lines

There are three possible ways to do this.

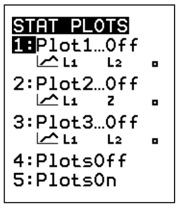
1. **Series of points:** First you need to put the x values of the points in one list, the y values in another. The standard lists on the TI-84 are L1 – L6 (2nd functions on the corresponding number keys), but you can also give lists names (up to five characters) if you want.

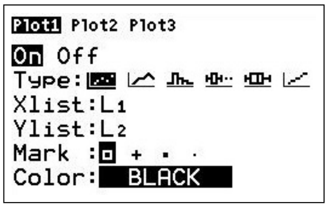
   The easiest way to enter the data is to press stat and select Edit.* Enter the x values in L1, pressing enter after each value including the last. Similarly, put the y values in L2.



   *If you don't see columns for L1, and L2, press stat 5 enter to reset the list editor. To clear a list, move the cursor to the list header and press clear.

   Now you need to tell the calculator what type of plot we want, where the data are and the type of marker for each point. Press 2nd y= to access the statplot menu.
   Up to three plots can be displayed. Press 1 to select Plot1.



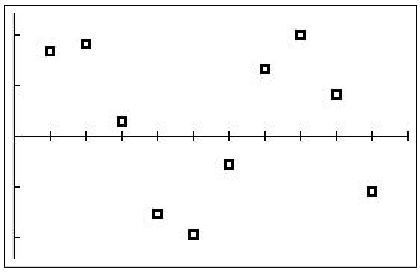


   By moving the cursor and pressing enter, select the following options in the menu:
   – On: turn the plot on;
   – the type of plot: the first is a scatterplot (just the points), the second a line plot (successive points joined by a line);
   – specify the lists (probably already L1 and L2);
   – select the marker you want; and
   – on a TI-84CE, the marker colour you want.

   Now press y=. Plot1 should be highlighted, indicating it will be plotted when you press graph.

   Turn off any functions, unless you want them plotted too. You can turn plots on and off here in the same way.

   Next the window. Either enter suitable values using window, then press graph, or use zoom 9 (ZoomStat), which sets a window and plots the graph.



window $[0, 11, 1] \times [–1.2, 1.2, 0.5]$

2. **Line segment:** use the *Line* command in the [draw] DRAW menu ([prgm] key).

   From the Home screen: Line (X1, Y1, X2, Y2). *Example*: Line (3, 4, 5, 2) draws the line (segment) between (3, 4) and (5, 2).

   To erase a line segment, use the *Line* command with a fifth argument 0: Line (3, 4, 5, 2, 0) erases the above line. On a CE, you can also set a different colour (default blue), chosen from the [vars] COLOR menu and/or different linestyles (1–4; default 2 = thick[16]) as the optional sixth argument; e.g. Line (3, 4, 5, 2, BLACK, 1) gives a thin black line.

   On a graph: select *Line*, move the cursor to one end of the line segment you want and press [enter]. Move the cursor to the other end of the line and press [enter] again. Keep drawing as many line segments as you want using [enter] to begin and end each segment. Press [clear] to cancel the *Line* command. On a CE, you can change the colour and linestyle in the STYLE menu [F5].

3. **Point:** use the *Pt–On* command in the [draw] POINTS menu.

   From the Home screen: Pt–On (X, Y). *Example*: Pt–On (3, 4) plots the point (3, 4). There are optional arguments for marker type (1–4) and, on a CE, colour, chosen from the [vars] COLOR menu.

   On a graph: select *Pt–On*, move the cursor to the point(s) you want and press [enter]. Press [clear] to cancel the *Pt–On* command. On a CE, you can change the marker type and colour in the STYLE menu [F5].

[16] The options here are: 1 thin line; 2 thick line; 3 shaded above; 4 shaded below.

3.3 Activities

3.3.1 Linear models

Solutions in Section 3.4.

Budgeting or not

The underlined amounts can be changed to whatever you like for individual assignments.

Brock has $100 in the bank, has no income, but is spending about $5 a week on his new girlfriend Amber.

His sister Minerva has only $10 in the bank, but is spending nothing and saving about $3 a week.

(a) Find the equation of the line that gives how much money each person has in the bank as a function of time in weeks. Graph these equations using suitable axes.

(b) What is the slope of each line: from the graphs? from the equations?

(c) What is the physical interpretation of the slope?

(d) What is the y intercept of each line: from the graphs? from the equations?

(e) What is the physical interpretation of the y intercept?

(f) From the graphs, when will Minerva and Brock have the same amount of money? Check your answer using the equations.

(g) From the equations, when will Minerva have twice as much money as Brock? How could you do this graphically?

Renting a car

The Rent-a-Wreck Car Rental Company has the cheapest car rentals in town.

You can choose one of two options.

- **Option A** — no flat fee, but a charge of 28c per kilometre.
- **Option B** — a flat rate of $36 per day, plus 18c per kilometre.

(a) You want to hire a car for one day.

- **(i)** Under Option A, how much will it cost in dollars to drive 1 km? 2 km? 10 km? 100 km? x km?
- **(ii)** For Option A, what is the equation for y, the cost in dollars, in terms of x, the number of kilometres driven?
  Check your equation with the numbers you calculated in (i).
- **(iii)** Under Option B, how much will it cost in dollars to drive 1 km? 2 km? 10 km? 100 km? x km?
- **(iv)** For Option B, what is the equation for y, the cost in dollars, in terms of x, the number of kilometres driven?
  Check your equation with the numbers you calculated in (iii).

(b) Graph the equations for the two options, if you will drive up to 600 km in the day.

(c) Estimate from the graph how far you have to drive before Option B becomes cheaper.

(d) From the graph, what is the slope of the line for Option A?
Hint: Press [trace] and use the left/right arrow keys to find two points on the graph; use these work out the slope.

What is the slope of the line for Option B?

What does the slope represent in this problem?

(e) What is the y intercept of the graph of Option A?
Hint: Press [trace], type in 0 and press [enter].

What is the y intercept of the graph of Option B?

What does the y intercept represent in this problem?

(f) Work out exactly how far you have to drive before Option B becomes cheaper.

Marketing a computer game

You have just written a fantastic computer game and your company wants to sell it. *How much should it charge?*

If it puts on a high price, the company won't sell as many games, but it will make more money per game sold. If the game is sold for a low price, the company won't make as much money on each game sold, but it will sell more games.

Clearly, the number sold depends on the price. Economists often assume that the number sold and price are related by a *linear equation.*

After doing some market research, the company thinks that if it sells the game for \$160 per copy, it will sell about 800 copies. If the price is dropped to \$40 per copy, it should sell about 8000 copies.

(a) Let's use a graph of number sold versus price to help us in our problem. Price will be on the x axis and number sold on the y axis. What are suitable scales for the axes?

(b) What is the slope of the line? *Hint:* Use the two points that we know on the graph of number sold versus price.

(c) What is the equation of the line? What extra information about the line do we use here other than its slope? Graph the line and check that the points you know actually lie on the graph of the line.

(d) Does this graph tell us the answer to the question of what the price should be?

(e) Revenue means total income. It is the product of price and number sold. Write down the equation for revenue as a function of price x. What kind of function is this?

(f) Plot the graph of revenue versus price. Estimate the revenue if the price is \$50.

(g) What is the best price to sell the game at? Why is it best? What is the revenue?

(h) What is the revenue if the game is sold at a price of \$180? Explain.

Acknowledgement to material from an unknown website.

3.3.2 Other activities

The following Coordinate Geometry activities are collected in the *Volume 1 Supplement: Activities for Years 9 and 10.* Year levels and subject matter are indicated with each summary. Solutions and teachers' notes are provided with each activity.

A Classic Problem — The Hare and Tortoise
The graphs of the distances covered versus time in this classic race are used to answer various questions about the race, such as who won and by how much. A fun exercise in putting questions into maths and solving equations graphically.
Year 10, Level 1; Algebra; Sketching Other Graphs, Simultaneous Equations.

Alien Attack
Uses one of Newton's equations of motion to explore properties of quadratic equations both numerically and graphically.
Years 9 and 10, Levels 1 and 2; Algebra; Coordinate Geometry.

Best Shape for a Can
Minimising the surface area of a cylinder (can) for a fixed volume. Numerical and graphical techniques, rather than Calculus, are used to find the minimum. Aspects of mathematical modelling are introduced.
Year 10, Level 1; Algebra/Measurement; Sketching Other Graphs/Volume.

Coordinate Geometry Art
A simple picture consisting of straight-line segments is 'coded' using the coordinates of the points of its vertices. These are used 'transmit' the picture to someone else. A graphics calculators is used to 'decode' and check the 'transmitted' picture.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

Let's Be Rational
Understanding the local and global behaviour of rational functions.
Year 10, Level 1; Algebra; Sketching Other Graphs, Polynomials.

Parabolic Aerobics
The first activity investigates the effect of changing the numbers A, B and C on the graphs of the family of parabolas $\mathrm{Y = A(X{-}B)^2{+}C}$. In the second activity, you must guess the numbers A, B and C for the graph of a mystery parabola generated by the PARABOLA/PRBOLACE program. The calculator checks your answers and keeps score.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

Probably Finding π
An experimental-probability method for finding π.
Year 10, Levels 1 and 2; Chance and Data; Probability.

Reaction Times and Statistics
Programs are used to measure reaction times in various scenarios. The data are displayed as box-and-whisker plots for subsequent analysis.
Years 8–10, Levels 1 and 2; Chance and Data; Statistics.

Simultaneous Equations
Solving simple simultaneous linear equations numerically (with a table), graphically and algebraically.
Year 9, Levels 1 and 2; Algebra; Solve Simultaneous Linear Equations.

Speeding — A Study in Linear Functions
Students apply basic knowledge of linear functions to problems involving speeding tickets.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

Starburst
A study of straight lines: slope and intercept.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

Statistics from Birthdays
Class data on day and month of birth are used to provide an introduction to data presentation on a graphics calculator.
Year 9, Levels 1 and 2; Chance and Data; Statistics.

Temperature Conversions
Application of linear functions to conversion between degrees Celsius and degrees Fahrenheit.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

Which Fuel?
Application of linear functions to choosing whether to use petrol or LPG in your car.
Years 9 and 10, Level 1; Algebra; Coordinate Geometry.

Programs

Some of these activities use programs; a list is given below. They are available at *www.XXX*. Details of using the programs are given in the relevant activity. For information on copying programs, see the last chapter in Volume 2 of this book.

CARSTOP/CARSTPCE — measures reaction times in simulating braking in a car. Used in *Reaction Times and Statistics*.

GUESSLIN/GSSLINCE — guess equation of graphed straight line. Used in *Guess My Line*.

PARABOLA/PRBOLACE — guess equation of a graphed parabola. Used in *Parabolic Aerobics*.

REACT/REACTCE — reaction times. Used in *Reaction Times and Statistics*.

REACTHND/RCTHNDCE — reaction times. Used in *Reaction Times and Statistics*.

SLIME/SLIMECE — simulates a projectile shot straight up. Used in *Alien Attack*.

3.4 Solutions

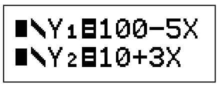
Solutions to the three activities in Section 3.3.1.

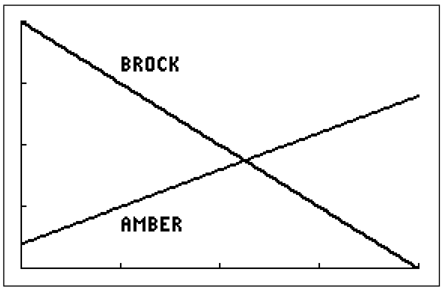
Budgeting or not

Brock has \$100 in the bank, has no income, but is spending about \$5 a week on his new girlfriend Amber. His sister Minerva has only \$10 in the bank, but is spending nothing and saving about \$3 a week.

(a) Find the equation of the line that gives how much money each person has in the bank as a function of time in weeks. Graph these equations using suitable axes.

If $y_{\text{B}}(t)$ is the amount of money Brock has in the bank after t weeks and $y_{\text{M}}(t)$ the corresponding amount for Minerva, $y_{\text{B}} = 100 - 5t$ and $y_{\text{M}} = 10 + 3t$. These equations are graphed in the figure below.





window $[0, 20, 5] \times [0, 100, 25]$

(b) What is the slope of each line: from the graphs? from the equations?

From the graphs, choose two points on each line.

From the graph of y_{B}, two points are $(0, 100)$ and $(20, 0)$, giving a slope $(100–0)/(0–20) = -5$.

From the graph of y_{M}, two points are $(0, 10)$ and $(20, 70)$, giving a slope $(10–70)/(0–20) = 3$.

From the equations, the slope is the coefficient of t: the slope of the graph of y_{B} is therefore -5; the slope of the graph of y_{M} is 3.

(c) What is the physical interpretation of the slope?

The slope is the (constant) change in the amount of money (\$) in the bank per week.

(d) What is the y intercept of each line: from the graphs? from the equations?

From the graphs, the y intercept of the graph of y_{B} is 100, the y intercept of the graph of y_{M} is 10.

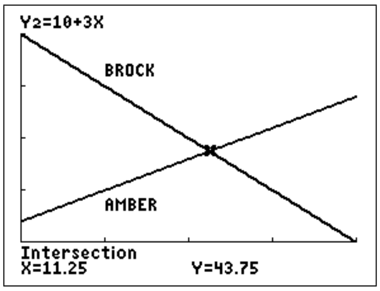
From the equations, the y intercept is the value of y when $x = 0$, giving the same answers.

(e) What is the physical interpretation of the y intercept?

The y intercept is the initial ($t = 0$) amount of money in the bank.

(f) From the graphs, when will Minerva and Brock have the same amount of money? Check your answer using the equations.

From the graphs, Minerva and Brock will have the same amount of money at the t value of the intersection of the two graphs. Using *intersect* in the calc menu (figure below) gives $t = 11.25$.



window $[0, 20, 5] \times [0, 100, 25]$

From the equations, we must solve $y_B = y_M$ for time t.

Therefore, $100 - 5t = 10 + 3t$.

Therefore, $8t = 90$, so that $t = 11.25$.

Minerva and Brock will have the same amount of money, \$43.75, between Week 11 and Week 12.

(g) From the equations, when will Minerva have twice as much money as Brock? How could you do this graphically?

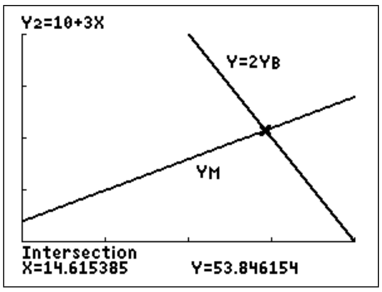
We must solve $y_M = 2y_B$ for time t.

Therefore, $10 + 3t = 2(100 - 5t)$.

Therefore, $13t = 190$, so that $t = 14.6$ to 1 decimal place.

Minerva will have twice as much money as Brock between Week 14 and Week 15.

Graphically, we could plot $y = 2y_B = 2(100–5t)$ and find its intersection with y_M (figure below), giving the same answer.



window $[0, 20, 5] \times [0, 100, 25]$

Renting a Car

The Rent-a-Wreck Car Rental Company has the cheapest car rentals in town.

You can choose one of two options.

- **Option A** — no flat fee, but a charge of 28c per kilometre.
- **Option B** — a flat rate of $36 per day, plus 18c per kilometre.

(a) You want to hire a car for one day.

(i) Under Option A, how much will it cost in dollars to drive 1 km? 2 km? 10 km? 100 km? x km?

0.28; 0.56; 2.80; 28.00; $0.28x$.

(ii) For Option A, what is the equation for y, the cost in dollars, in terms of x, the number of kilometres driven?
Check your equation with the numbers you calculated in (i).

$y_1 = 0.28x$

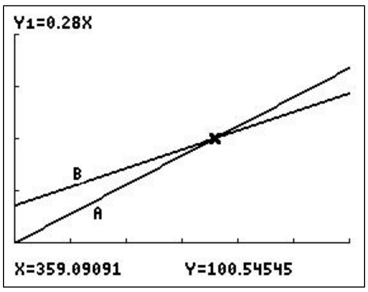
(iii) Under Option B, how much will it cost in dollars to drive 1 km? 2 km? 10 km? 100 km? x km?

36.18; 36.36; 37.80; 54.00; $36+0.18x$.

(iv) For Option B, what is the equation for y, the cost in dollars in terms of x, the number of kilometres driven?
Check your equation with the numbers you calculated in (iii).

$y_2 = 36 + 0.18x$

(b) Graph the equations for the two options, assuming you will drive up to 600 km in the day.



window $[0, 600, 100] \times [0, 200, 50]$

(c) Estimate from the graph how far you have to drive before Option B becomes cheaper.

Using trace, you have to drive somewhere between 359 km and 364 km (adjacent trace pixels). We could use *intersect* to find this more accurately but we do it exactly in (f).

(d) From the graph, what is the slope of the line for Option A?

Using trace, we find points $(100, 28)$ and $(400, 112)$ to be on the graph.

Therefore, slope $= \dfrac{112 - 28}{400 - 100} = \dfrac{74}{300} = 0.28.$

Of course, we could have seen this directly from the equation for Option A.

What is the slope of the line for Option B?

Using trace again, we find points $(100, 54)$ and $(400, 108)$ to be on the graph.

Therefore, slope $= \dfrac{108 - 54}{400 - 100} = \dfrac{54}{300} = 0.18$.

Again, we could have seen this directly from the equation for Option B.

What does the slope represent in this problem?

The slope represents the cost of driving in dollars per kilometre.

(e) What is the y intercept of the graph of Option A? $y = 0$

What is the y intercept of the graph of Option B? $y = 36$

What does the y intercept represent in this problem?

The initial cost (before driving the car).

(f) Work out exactly how far you have to drive before Option B becomes cheaper.

We have to find out at what x value $y_1 = y_2$, that is $0.28x = 36 + 0.18x$, with solution $x = 360$.

You have to drive 360 km before Option B becomes cheaper, consistent with our answer to (c).

Marketing a Computer Game

You have just written a fantastic computer game and your company wants to sell it. *How much should it charge?*

If it puts on a high price, the company won't sell as many games, but it will make more money per game sold. If the game is sold for a low price, the company won't make as much money on each game sold, but it will sell more games.

Clearly, the number sold depends on the price. Economists often assume that the number sold and price are related by a *linear equation.*

After doing some market research, the company thinks that if it sells the game for $160 per copy, it will sell about 800 copies. If the price is dropped to $40 per copy, it should sell about 8000 copies.

(a) Let's use a graph of number sold versus price to help us in our problem. Price will be on the x axis and number sold on the y axis. What are suitable scales for the two axes?

$0 < x < 160$; $0 < y < 8000$

(b) What is the slope of the line?

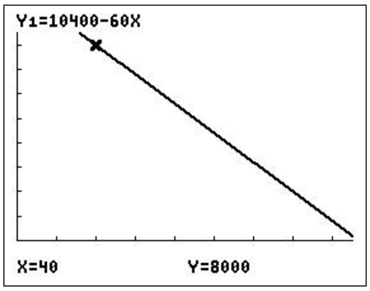
The two points we have are $(160, 800)$ and $(40, 8000)$.

The slope of the line is $\dfrac{8000 - 800}{40 - 160} = -60$.

(c) What is the equation of the line? What extra information about the line do we use here other than its slope? Graph the line and check that the points you know actually lie on the graph of the line.

The equation of the line is of the form $y = mx + b$, where m is the slope and b is a constant determined by a point on the line, the extra information. Setting $m = -60$ and taking the point $(160, 800)$, we have $800 = -60 \times 160 + b$, so that $b = 800 + 60 \times 160 = 10{,}400$.

The equation of the line is therefore $y = 10{,}400 - 60x$.



window $[0, 170, 20] \times [0, 8500, 1000]$

Using trace confirms that the two points we know lie on the graph of the line. One of these is shown in the figure.

(d) Does this graph tell us the answer to the question of what the price should be?

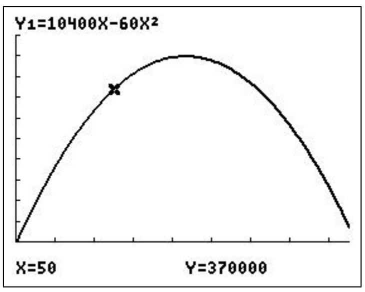
No. There is no clear optimum value.

(e) Revenue means total income. It is the product of price and number sold. Write down the equation for revenue as a function of price x. What kind of function is this?

From our equation above, revenue $R = xy = x(10{,}400 - 60x) = 10{,}400x - 60x^2$.

This is a quadratic function.

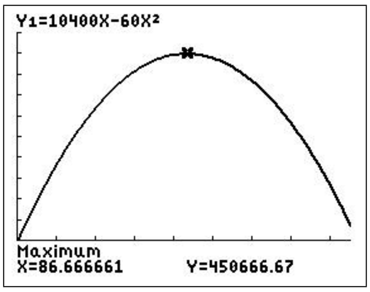
(f) Plot the graph of revenue versus price. Estimate the revenue if the price is \$50.



window $[0, 170, 20] \times [0, 5 \times 10^5, 5 \times 10^4]$

If the price is \$50, the revenue is \$370,000 (see the figure).

(g) What is the best price to sell the game at? Why is it best? What is the revenue?



window $[0, 170, 20] \times [0, 5 \times 10^5, 5 \times 10^4]$

The best price to sell the game at is the price that maximises revenue; from the graph using *maximum*, that price is \$86.67 or \$87 to the nearest dollar. The corresponding revenue is \$450,667 to the nearest dollar.

(h) What is the revenue if the game is sold at a price of \$180? Explain.

The revenue predicted by the model if the game is sold at a price of \$180 is $-\$7,000$. Clearly a negative revenue does not make sense; the price of \$180 lies outside the range of the model.

4 Inequalities and Linear Programming — Numerical, Graphical and Algebraic Approaches

4.1 Linear inequalities

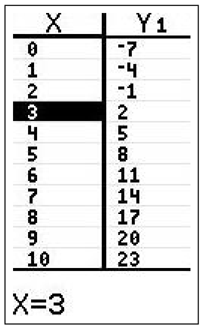
Linear inequalities can be solved numerically, graphically or algebraically. For simple linear inequalities, the algebraic method (manipulating symbols) is probably the quickest but once the inequalities become more complicated, the graphical method comes into its own. Some may prefer the numerical method but having the big picture of a graph, particularly with the graphics tools in the calc menu, is always useful.

We start with some simple inequalities to demonstrate the three methods before moving on to more complicated examples.

Example 1: Find all values of x for which $3x-7\leqslant 2$.

Numerical method

- Press y= and enter the left-hand side of the inequality into Y_1.
- Press tblset (2nd window) and set the values as shown. The table will start at $x=0$ and increment in steps of 1.
- Press table.
- Here scroll in the X column if necessary to find where $Y_1\leqslant 2$.
- You should find that $Y_1\leqslant 2$ when $x\leqslant 3$.
- Therefore, $3x-7\leqslant 2$ when $x\leqslant 3$.

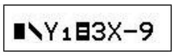


How do we know that there are not some other values of $x\leqslant 3$ at which $3x-7>2$ or values of $x>3$ at which $3x-7\leqslant 2$?

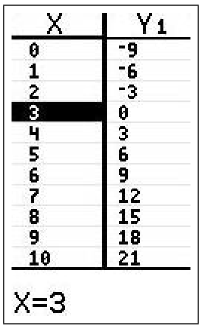
We don't from the table, no matter how much we scroll up or down. It is most obvious graphically (over the page) but can only be *proved* algebraically.

PTO

Sometimes it is easier to see when a function is greater than or less than 0, rather than comparing values of the two functions.



- Rewrite the inequality as $3x-9\leqslant 0$.
- Set Y1 = 3X−9.
- Press [table] to see where Y1⩽0. Verify the solution for the inequality that you found above.

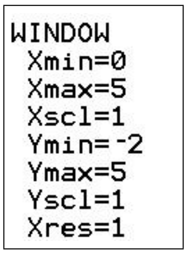


X	Y1
0	-9
1	-6
2	-3
3	0
4	3
5	6
6	9
7	12
8	15
9	18
10	21

X=3

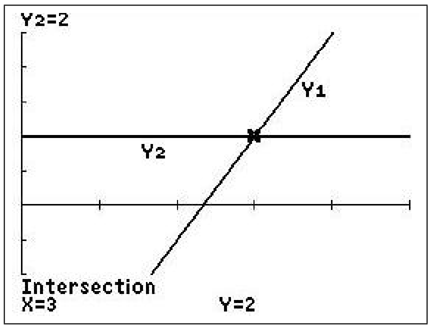
Graphical method

- Set Y1 = 3X−7 again. Set Y2 = 2, an appropriate [window] and press [trace]. This gives two straight lines which intersect at only one point.



- We must find the region (in terms of x) for which points on Y1 have smaller y values than the corresponding points on Y2.

  Find the point of intersection of the two lines, the point at which the y values are equal, by pressing [calc] ([2nd] [trace]) [5] (*intersect*), [enter] twice to select the lines you are finding the intersection of, moving the cursor to provide a rough *Guess* and pressing [enter].



- Observe that for all values of x less than or equal to 3, the left-hand side of the inequality (Y1) is less or equal to than the right-hand side (Y2).
- Therefore, the solution is $x\leqslant 3$.

Algebraic method

Here we manipulate symbols in the same way as when we want to isolate a variable in an equation. However, keep in mind that if you divide or multiply an inequality by a negative number, the inequality sign must be reversed. For this reason, you should never multiply or divide by a variable when solving inequalities.

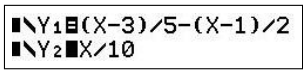
$$3x-7 \leqslant 2$$

- Add 7 to both sides: $3x \leqslant 9$.
- Divide by 3 (positive): $x \leqslant 3$.
- Therefore, the only solution is $x \leqslant 3$.

Example 2: Find all values of x for which $\dfrac{x-3}{5} - \dfrac{x-1}{2} \leqslant \dfrac{x}{10}$.

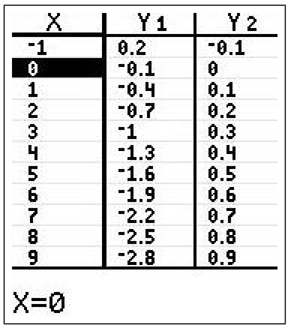
Numerical method

- Press [y=] and enter the left-hand side of the inequality into Y1 and the right-hand side into Y2.

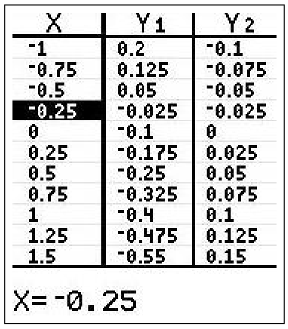


- Press [tblset] ([2nd] [window]) and set the values as shown. The table will start at $x=0$ and increment in steps of 1.

- Press [table].
- Here scroll up in the X column to find when Y1 is less or equal to than Y2.

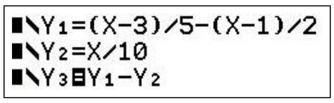


- When you have narrowed down the point at which the two are equal to between two integers (here between −1 and 0), press [tblset] again, set *TblStart* to the smaller of the two integers (−1) and Δ*Tbl* to 0.25. Press [table] again.
- Repeat again as necessary with new *TblStart* and smaller Δ*Tbl* to solve the inequality. You should find that Y1⩽Y2 when $x \geqslant -0.25$.
- Therefore, the solution is $x \geqslant -0.25$.

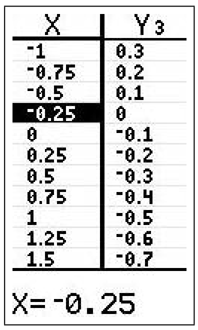


Again, it is sometimes easier to see when a function is greater than or less than 0, rather than comparing values of the two functions.

- Rewrite the inequality as
$$\frac{x-3}{5} - \frac{x-1}{2} - \frac{x}{10} \leqslant 0,$$
or, in calculator terms Y1 − Y2 ⩽ 0.

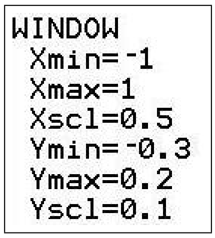


- Set Y3 = Y1 − Y2 ([vars] Y-VARS Function).
Turn off Y1 and Y2 by moving the cursor over the = sign and pressing [enter].
- Press [table] to see the values of Y3. If necessary, proceed as we did above to narrow down the solution. Verify the solution for the inequality that you found above.



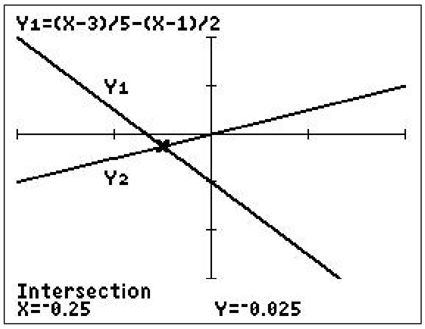
Graphical method

- The two sides of the inequality are already entered into Y1 and Y2. Turn these two functions back on, turn off or delete Y3, set an appropriate [window] and press [trace].



- We must find the region (in terms of x) for which points on Y1 have smaller y values than the corresponding points on Y2.

  Find the point of intersection of the two lines, the point at which the y values are equal, by pressing [calc] ([2nd] [trace]) [5] (*intersect*), [enter] twice to select the lines you are finding the intersection of, moving the cursor to provide a rough *Guess* and pressing [enter].



- Observe that for all values of x greater than -0.25, the left-hand side of the inequality (Y1) is less than the right-hand side (Y2).
- Therefore, the only solution is $x \geqslant -0.25$.

Algebraic method

$$\frac{x-3}{5} - \frac{x-1}{2} \leqslant \frac{x}{10}$$

- Multiply both sides by 10 to clear the fractions: $\frac{10(x-3)}{5} - \frac{10(x-1)}{2} \leqslant \frac{10x}{10}$.
- Cancel out numerical factors: $2(x-3)-5(x-1) \leqslant x$.
- Expand brackets and simplify: $-3x-1 \leqslant x$.
- Add $3x$ to both sides: $-1 \leqslant 4x$.
- Divide by 4: $-\frac{1}{4} \leqslant x$ or $x \geqslant -\frac{1}{4}$.
- Therefore, the only solution is $x \geqslant -0.25$.

Geometrically, linear inequalities give two straight lines. Two straight lines can either not intersect (parallel lines), giving no solution, intersect at a point, giving one solution, or are the same line. The latter gives an infinite number of solutions if the inequality is $\leqslant$ or $\geqslant$ but no solutions if the inequality is $<$ or $>$.

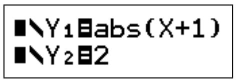
4.2 Absolute-value inequalities

The same three methods can also be used with absolute-value inequalities. Recall that the absolute value of x or mod x, written $|x|$, is **defined** as

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

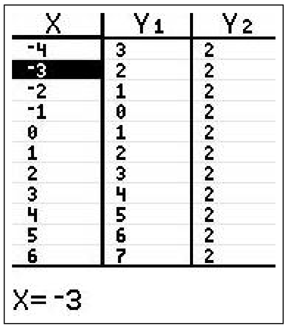
Example 3: Solve the inequality $|x+1| < 2$.

Numerical method



- Press [y=]: set Y1 = abs(X+1) and Y2 = 2.

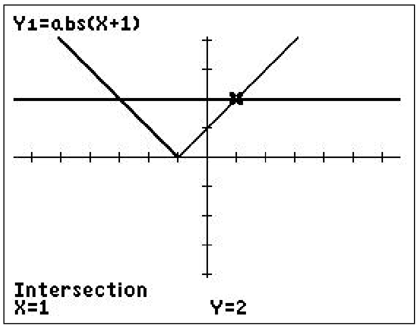
  abs is in the [math] NUM menu.
- Press [tblset]: set *TblStart* = 0, Δ*Tbl* = 1.
- Press [table].
- Scroll up and down in the X column to find when Y1 < Y2: $-3 < x < 1$.
- Alternatively, set Y1 = abs(X+1)−2 and use [table] to find where Y1 < 0.



Graphical method $\quad |x+1| < 2$

- Put the functions in Y1 and Y2 as described in *Numerical method*.
- Press [zoom] [4] to set the window and graph the function. Note the values of the cursor coordinates — the result of [zoom] [4].
- Find the two intersection points using *intersect*, as described on page 42.
- From the points of intersection we see that, for all points to the right of -3 and to the left of 1, the graph of $y = |x+2|$ has $y < 2$.

  Therefore, the solution is $-3 < x < 1$.



As seen in the Algebraic method (over the page), and shown here, this absolute inequality actually corresonds to two separate inequalities (one straight line becomes two), allowing the possibility of 0, 1 or 2 solutions.

PTO

Algebraic method $|x+1| < 2$

From the definition of $|x|$, the inequality $|x-b| < a$ means that[17]

$$x-b < a \quad \textbf{and} \quad -(x-b) < a.$$

Here, we have $a=2$ and $b=-1$, so $|x+1| < 2$ means that

$$x+1 < 2 \quad \textbf{and} \quad -(x+1) < 2.$$

Multiplying the second inequality by -1 and therefore changing the inequality direction gives

$$x+1 < 2 \quad \textbf{and} \quad x+1 > -2,$$

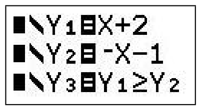
so that $x<1$ **and** $x>-3$. These can be combined to give $-3<x<1$.

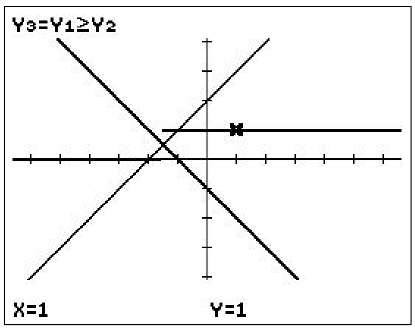
4.3 Using logic functions to solve inequalities

An interesting way to solve inequalities graphically is to use logic functions or relational operators.

Example 4: Solve the inequality $x+2 \geqslant -x-1$.

- Set Y1 = X+2 and Y2 = −X−1. Watch − signs here.
- Set a window ([zoom] [4] in the figure below) and graph the the lines. Adjust the window until the intersection point of the lines lies somewhere near the middle of the screen.
- Set Y3 = Y1 ≥ Y2. The ≥ symbol is in the [test] ([2nd] [math]) menu.
- Press [graph] and a new line will now appear on the graph.[a]
- Press [trace] and the down arrow until you select the new line. Move the cursor along it and you will see that the line has y values of 1 or 0. If $y = 0$, the corresponding value of x does not satisfy the inequality; if $y=1$, it does.
- We can see that for all values of x greater than -1.5, the y value is 1, so the solution must be $x \geqslant -1.5$.





[a] If there is an almost vertical line at the start of the horizontal line: on a TI-84CE, press [format] and turn on *Detect Asymptotes*; on a TI-84, choose a dotted line for Y3 in [Y=]. Press [graph] to redraw the functions.

Conclusion: The algebraic method is quickest for simple linear inequalities, the graphical method for more-complicated inequalities (see Section 4.8). As in most mathematics, a diagram (graph) is always useful.

[17] Alternatively, $-a < x - b < a$.

4.4 Graphing regions

The line $ax+by+c=0$ divides the number plane into two regions (known as half-planes). All the points that satisfy the inequality $ax+by+c<0$ lie on one side of the line, while all the points that satisfy the opposite inequality $ax+by+c>0$ lie on the other side. When given such an inequality to solve, the answer can be shown by shading the appropriate region on a graph.

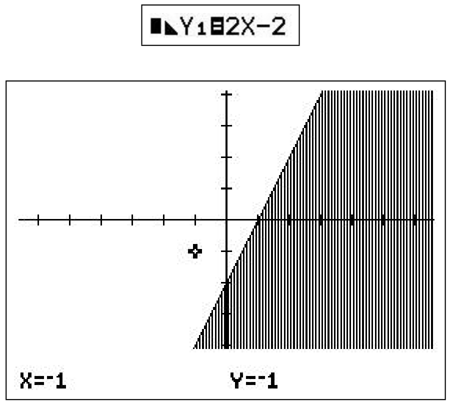
The inequality has to be rewritten with y on one side, so it can be entered into the calculator.

Example 5: Graph the region satisfying the inequality $2 < 2x-y$ for $-10<x<10$.

- Rewrite the inequality as $y < 2x-2$.
- Press [y=] : set $Y_1 = 2X-2$.
- Move the cursor over the slanted line to the left of Y_1 and press [enter].

  Toggle through the various line styles until the section below the line is shaded. This selects the graph style in which the area under Y_1 is shaded, corresponding to y values less than those on the line $y=2x-2$.
- Press [zoom] [4] to set a viewing window and graph the function.

  In general, you will have to experiment with the window parameters to find the best settings for any given inequality.



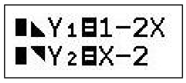
Move the cursor around the screen with the arrow keys to verify that y is less than $2x-2$ in the shaded region, but not in the unshaded region. The boundary line here, $y=2x-2$, is not in the region because of the $<$ sign.

PTO

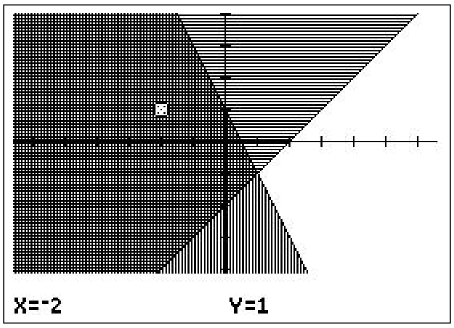
4.5 Compound inequalities

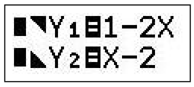
While a single inequality divides the number plane into two regions when graphed, two inequalities graphed simultaneously divide the number plane into three or more regions. The graphics calculator can be used to shade the regions and thus see if there is a region which satisfies both the inequalities.

Example 6: Find the solution region to the inequalities $2x+y<1$ **and** $y>x-2$.

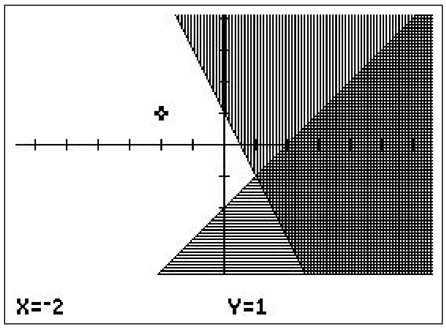


- First rearrange $2x+y<1$ as $y<1-2x$.
- Enter 1−2X into Y1 and select shading *below* (a lesser-than inequality); enter X−2 into Y2 and shading *above* (a greater-than inequality).
- Press [zoom] [4].
- The area in which both inequalities are shaded (the hatched region) is the solution region: any values/points (x, y) lying in that region satisfy both inequalities, as you can verify by moving the cursor around and substituting its coordinates into the inequality.





- Sometimes it is useful to have the region we are interested in unshaded, rather than double shaded, as above.
- To do this, simply reverse the shading chosen for each function on the calculator.
- In the example above, select shading *above* for Y1 and shading *below* for Y2.
- The resulting graph has the region satisfying both inequalities unshaded.



4.6 Exercises

Answers are provided here, full solutions in Section 4.9.1.

In 1−4, solve the following inequalities using any or all of the three approaches.

1. $2+5x \geqslant 1$. $(x \geqslant -0.2)$

2. $5+4x > 2x+1$. $(x > -2)$

3. $|2-5x| \leqslant 3$. $(-0.2 \leqslant x \leqslant 1)$

4. $|2x-3| \leqslant |2-x|$. $\left(1 \leqslant x \leqslant \frac{5}{3}\right)$

5. Graph the region satisfying the inequality $1 < x+2y$ for $-10<x<10$.

6. Graph the region satisfying the inequalities $x-3y<2$ **and** $y>2x-1$.

4.7 Linear Programming

4.7.1 Method

Inequalities are particularly useful with problems of allocation, in which there is a region of possible answers and you must select the most effective or profitable one. In its simplest form, this is called Linear Programming.

Graphics are essential here, both for understanding the process and doing the problems. The graphs can be done by hand, and should be once or twice, but this becomes rather boring. Graphics calculators allow one to concentrate on the process. The method is illustrated by the following example.

Example 7: Two foods provide the main source of carbohydrate and protein for a diet. Food A costs \$4 per kilogram and Food B costs \$5 per kilogram. What is the cheapest mix of the two foods that provides enough protein and carbohydrate, given the following information?

- Each kilogram of Food A contains 10 units of protein and 30 units of carbohydrate.
- Each kilogram of Food B contains 20 units of protein and 20 units of carbohydrate.
- Daily requirements: protein — at least 30 units; carbohydrate — at least 60 units.

Let the amount of Food A you eat be x kg and the amount of Food B y kg. The constraints can then be written

$10x+20y \geqslant 30$ amount of protein

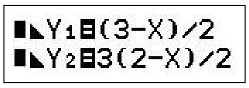
$30x+20y \geqslant 60$ amount of carbohydrate

$x,\ y \geqslant 0$ physical constraints (you can't have negative amounts of food).

Arrange these so that y is the subject.

$$y \geqslant (30-10x)/20 = (3-x)/2$$

$$y \geqslant (60-30x)/20 = 3(2-x)/2$$

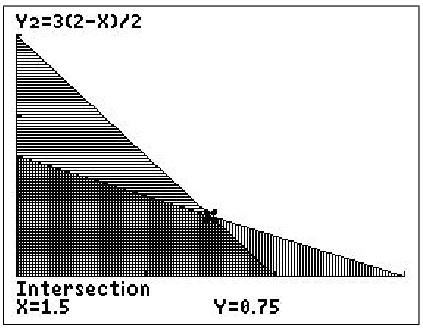


Enter the right-hand sides of these inequalities into Y1 and Y2 respectively.

Normally we would select *greater-than* shading, but here it is useful to have the region of interest unshaded. Therefore, select *less-than* shading or shading under for Y1 and Y2.

Set an appropriate window (note the third constraints) and press graph.

The two curves intersect at the point $(1.5, 0.75)$.



window $[0,3,1]\times[0,3,1]$

Any point in the unshaded area provides enough protein and carbohydrate. In order to determine which of the possible solutions (which point in the unshaded area) is best, we need to create a cost line for our model.

Food A costs \$4 per kilogram and Food B \$5 per kilogram. Therefore, the total cost C (\$) is $4 \times$ amount of Food A $+\ 5 \times$ amount of Food B, i.e.

$$C = 4x + 5y.$$

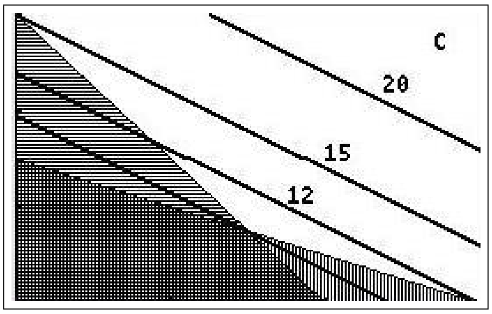
Rewriting this expression to make y the subject, we obtain the equation of the cost line

$$y = \frac{C}{5} - \frac{4}{5}x.$$

Each value of C gives a cost line — all points lying on a given cost line will give the same total cost. Moreover, all the cost lines are parallel: they all have a slope of $-4/5$. We want to find the cost line with the smallest C that still intersects the unshaded area.

Drawing in some lines with different values for C, it is clear that the further the lines are to the lower left, the smaller C and the lower the total cost.

The (unlabelled) cost line that passes through the point of intersection of the two lines (Y1 and Y2)[a] defining the unshaded region will therefore give the smallest C; this point is a vertex of the unshaded quadrilateral.



[a]Points on these lines satisfy the $\geqslant$ inequalities.

With the intersection point $(1.5, 0.75)$ (this point satisfies the $\geqslant$ inequalities),

$$C = 4 \times 1.5 + 5 \times 0.75 = 9.75.$$

Therefore, the cheapest way of getting your dietary requirements is to eat 1.5 kg of Food A and 0.75 kg of Food B per day, at a total cost of \$9.75.

The theory of Linear Programming says that the solution always lies at a vertex of the region created by the inequalities.

PTO

4.7.2 Activities

Answers are provided here, full solutions in Section 4.9.2.

The fruit basket

The produce manager of a grocery is making up fruit baskets to sell as gifts. They are to sell for no more than \$5, and contain only apples and oranges. She wants to get 24c per orange, 12c per apple and 68c for the basket. No more than 26 pieces of fruit will fit in the basket. Suppose she uses x oranges and y apples.

(a) Show that we are looking for those x and y values that satisfy the inequalities

$$\begin{aligned} x &\geqslant 0 \qquad & x+y &\leqslant 26 \\ y &\geqslant 0 \qquad & 2x+y &\leqslant 36 \end{aligned}$$

(b) When the equality signs are used we have the equations for four lines. Draw a diagram showing those four lines, then find an area of the xy plane in which useful x and y values must be found. Shetch that area. *Remember to explain what you are doing.*

(c) Which of these (x, y) values could the manager use?

$$(5, 10) \quad (10, 5) \quad (5, 25) \quad (20, 5) \quad (-2, 10)$$

(d) If she makes a profit of 3 cents on every orange sold and 2 cents on every apple sold, what is the equation for the total profit P? Draw in the $P = 30$ and $P = 45$ lines on your diagram. Can you see how to get the maximum profit? *Explain your reasoning.*

Answer: Maximum profit 62 cents; 10 oranges and 16 apples.

Market garden

The area available for crops on a farm is $1000\,\mathrm{m}^2$. Two crops grow well in the area and are being considered for planting. Each crop takes a different amount of time per m^2 to prepare the soil and plant. The total cost of this soil preparation and planting should not exceed \$2,500.

- Beans (per m^2): cost \$3; profit \$2.
- Spinach (per m^2): cost \$2; profit \$1.50.

What areas of each should be planted to maximise profit?

Answer: $500\,\mathrm{m}^2$ of each crop; profit \$1750.

Manufacturing #1

A factory manufactures doodads and whirligigs. It costs \$2 and takes 3 person hours to produce a doodad. It costs \$4 and takes 2 person hours to produce a whirligig. The factory has \$220 and 150 person hours a day to produce these products. If each doodad sells for \$6 and each whirligig sells for \$7, then how many of each product should be manufactured each day in order to maximise profit?

Answer: 20 doodads and 45 whirligigs; profit \$215.

Manufacturing #2

A manufacturer has 750 m of cotton fabric and 1000 m of polyester fabric. Production of a sweatshirt requires 1 m of the cotton and 2 m of the polyester, while production of a shirt requires 1.5 m of the cotton and 1 m of the polyester. The sale prices of a sweatshirt and a shirt are $30 and $24, respectively. How many of each type of shirt should be produced to maximise the return on sales (assuming all are sold)?

Answer: 375 sweatshirts and 250 shirts; return $17,250.

Brainbuilding

A student is looking for supplement protein bars to help his brain work faster, and there are two available products: protein bar A and protein bar B.

Each protein bar A contains 15 g of protein and 30 g of carbohydrate, and has total of 200 calories. Each protein bar B contains 30 g of protein and 20 g of carbohydrate, and has total of 240 calories.

According to his nutritional plan, this student needs at least 20,000 calories from these supplements over the month, which must comprise at least 1,800 g of protein and at least 2,200 g of carbohydrates.

If a protein bar A costs $3 and a protein bar B $4, what is the least possible amount of money he can spend to meet all his one-month requirements?

Answer: 70 of protein bar A and 25 of protein bar B; total cost $310.

Feeding the cows

A farmer feeds his cows a feed mix to supplement their foraging. The farmer uses two types of feed for the mix. Corn feed contains 100 g of protein per kg and 750 g of starch per kg. Wheat feed contains 150 g of protein per kg and 700 g of starch per kg. Each cow should be fed at most 7 kg of feed per day. The farmer would like each cow to receive at least 650 g of protein and 4000 g of starch per day. If corn feed costs $0.40/kg and wheat costs $0.45/kg, what is the feed mix that minimises cost? Round your answers to the nearest gram.

Answer: 3.412 kg of corn feed and 2.059 kg of wheat feed per cow; total cost $2.29.

PTO

Pizza profits

Modified from *Profiteering from Pizza* in *The Atomic Project*, V. Geiger, J. McKinlay and G. O'Brien (eds), AAMT, 1997.

A company produces two frozen pizzas, the Gluttono and the Carnivore. The three main ingredients used in both pizzas are cheese, tomato and meat. The quantity of these ingredients required for each pizza and the weekly supply of these are listed in the table below.

Ingredient	**Grams required per Gluttono**	**Grams required per Carnivore**	**Weekly supply** (kg)
cheese	70	80	96
tomato	60	40	70
meat	10	60	48

The company makes a profit of \$4 on each Gluttono and \$5.50 on each Carnivore.

(a) How many of each type of pizza should the company produce each week to maximise the profit?

Answer: 564 Gluttonos and 705 Carnivores.

(b) What quantity of ingredients, if any, are left unused when the maximum profit is generated?

Extensions

(c) If the amount of cheese available increases to 110 kg per week, what is the effect on production, given the company still strives to maximise its profit?

(d) What is the effect of a 10% decrease in the profit made on a Gluttono pizza?

(e) What change in profit on the Gluttono pizza would alter the optimal solution?

PTO

4.8 Quadratic inequalities

4.8.1 Methods

Example 8: Find all values of x for which $x^2-x-2<0$.

Numerical method

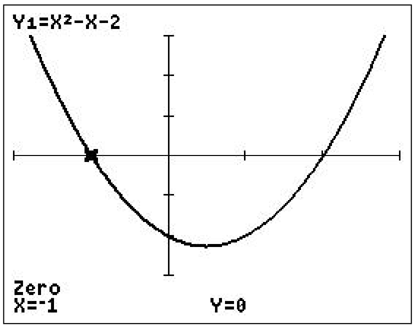
- Press y= and enter the left-hand side of the inequality into Y1.
- Press tblset : set $\mathit{TblStart}=0$, $\Delta \mathit{Tbl}=1$.
- Press table .
- Scroll up and down in the X column to find when Y1 <0: $-1<x<2$.

X	Y1
-2	4
-1	0
0	-2
1	-2
2	0
3	4
4	10
5	18
6	28
7	40
8	54

Graphical method

- Press y= and enter the left-hand side of the inequality into Y1.
- Turn off shading if necessary and set an appropriate window .
- Press graph to show the curve.
- Use *zero* in the calc menu to find the (two) points where the curve crosses the x axis.

  $x^2-x-2<0$ for $-1<x<2$.



window $[-2,3,1]\times[-3,3,1]$

Algebraic method

There will only be a solution here if the quadratic has two real roots.[18] This is the case if $b^2-4ac>0$ in the quadratic formula; here, $b^2-4ac=8>0$. To solve the inequality, we then need to factorise the quadratic: $x^2-x-2=(x+1)(x-2)$.

Then, $(x+1)(x-2)<0 \implies x+1<0$ **and** $x-2>0$ (1)

or

$x+1>0$ **and** $x-2<0$. (2)

Equation (1) gives $x<-1$ **and** $x>2$, which has no solution.

Equation (2) gives $x>-1$ **and** $x<2$, that is $-1<x<2$, which is the algebraic solution.

[18] If the inequality were $\leqslant$, it would be satisfied if there were only one real root: $b^2-4ac=0$.

Example 9: Find all values of x for which $x^2-x-2 \leqslant 4$.

Numerical method

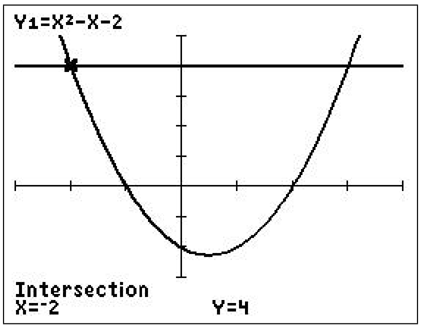
Press [y=] and enter the left-hand side of the inequality into Y1, the right-hand side into Y2. Proceed as in the previous example.
Alternatively, write the inequality as $x^2-x-6 \leqslant 0$.

$x^2-x-2 \leqslant 4$ for $-2 \leqslant x \leqslant 3$.

X	Y1	Y2
-3	10	4
-2	4	4
-1	0	4
0	-2	4
1	-2	4
2	0	4
3	4	4
4	10	4
5	18	4
6	28	4
7	40	4

Graphical method

- Press [y=] and enter the left-hand side of the inequality into Y1, the right-hand side into Y2.
- Set an appropriate [window].
- Press [graph] to show the curve.
- Use *intersect* in the [calc] menu to find the (two) points where the curve crosses $y=4$.
  $x^2-x-2 \leqslant 4$ for $-2 \leqslant x \leqslant 3$.



window $[-3, 4, 1] \times [-3, 5, 1]$

Algebraic method

Rewrite the inequality with 0 on the right-hand side: $x^2-x-6 \leqslant 0$.

There will only be a solution here if the quadratic has at least one real root. This is the case if $b^2-4ac \geqslant 0$ in the quadratic formula; here, $b^2-4ac = 25 > 0$.

To solve the inequality, we then need to factorise the quadratic: $x^2-x-6 = (x+2)(x-3)$.

Then, $(x+2)(x-3) \leqslant 0 \implies x+2 \leqslant 0$ **and** $x-3 \geqslant 0$ (1)

or

$x+2 \geqslant 0$ **and** $x-3 \leqslant 0$. (2)

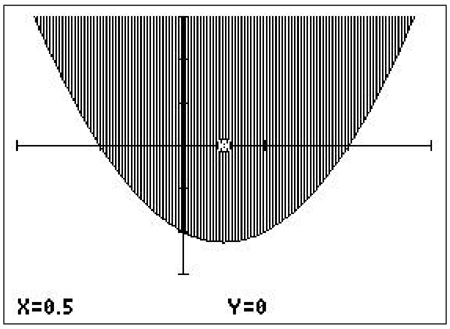
Equation (1) gives $x \leqslant -2$ **and** $x \geqslant 3$, which has no solution.

Equation (2) gives $x \geqslant -2$ **and** $x \leqslant 3$, that is $-2 \leqslant x \leqslant 3$, which is the algebraic solution.

PTO

Example 10: Sketch the region in the xy plane for which $y > x^2 - x - 2$.

- Press [y=] and enter the right-hand side of the inequality into Y1.
- Select shading above the function and set an appropriate [window]. You may need to experiment with the values.
- Press [graph] to show the region, the points in which satisfy the inequality.
- Pressing [trace] puts the cursor on the curve dividing the two regions. Here, because of the > sign, this curve $y = x^2 - x - 2$ does not lie in the region specified by the inequality.



window $[-2, 3, 1] \times [-3, 3, 1]$

4.8.2 Exercises

Answers are provided here, full solutions in Section 4.9.3.

Solve the following inequalities using any or all of the three approaches.

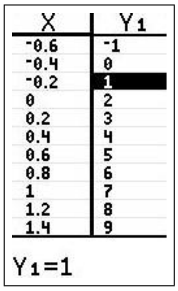
1. $x^2 - 2x - 3 \leqslant 0$. $(-1 \leqslant x \leqslant 3)$
2. $x^2 - 2x - 3 \leqslant 5$. $(-2 \leqslant x \leqslant 4)$
3. $20x^4 - 4x^3 - 143x^2 + 46x + 165 < 0$. $(-2.5 < x < 1$ or $1.5 < x < 2.2)$
4. $|x^2 - 2x - 3| \leqslant 2$. $(-1.449 \leqslant x \leqslant -0.4142$ or $2.414 \leqslant x \leqslant 3.449)$

4.9 Solutions

4.9.1 Exercises Section 4.6

1. Solve the inequality $2+5x \geqslant 1$ using the three approaches.

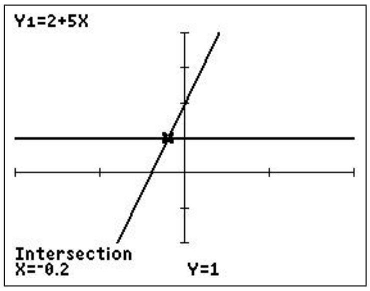
Numerically: proceeding as in the example, setting Y1 = 2+5X, we obtain the table shown in the figure with ΔTbl = 0.2. From this, we see that $2+5x \geqslant 1$ if $x \geqslant -0.2$.



X	Y1
-0.6	-1
-0.4	0
-0.2	1
0	2
0.2	3
0.4	4
0.6	5
0.8	6
1	7
1.2	8
1.4	9

Y1=1

Graphically: setting Y1 = 2+5X and Y2 = 1, we obtain the graph in the figure. Using *intersect*, we see that $2+5x \geqslant 1$ if $x \geqslant -0.2$.

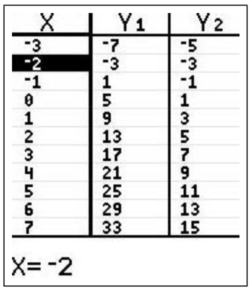


window $[-2,2,1]\times[-2,4,1]$

Algebraically: subtracting 2 from both sides of the inequality and dividing by 5, we have $x \geqslant -0.2$.

2. Solve the inequality $5+4x > 2x+1$ using the three approaches.

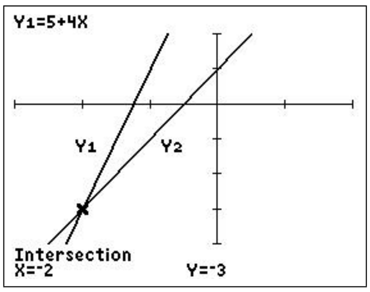
Numerically: setting Y1 = 5 + 4X and Y2 = 2X+1, we obtain the table shown in the figure with ΔTbl = 1. From this, we see that $5+4x > 2x+1$ (Y1>Y2) if $x > -2$.



X	Y1	Y2
-3	-7	-5
-2	-3	-3
-1	1	-1
0	5	1
1	9	3
2	13	5
3	17	7
4	21	9
5	25	11
6	29	13
7	33	15

X= -2

Graphically: setting Y1 = 5 + 4X and Y2 = 2X+1, we obtain the graph shown in the figure. Finding the intersection point of the two lines, we see that $2+5x > 2x+1$ (Y1>Y2) if $x > -2$.



window $[-3,2,1]\times[-4,2,1]$

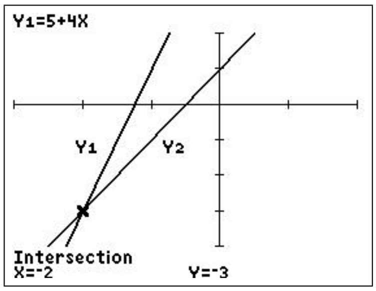
Algebraically: subtracting $2x$ and 5 from both sides of the inequality, we have $2x > -4$, so $x > -2$.

3. Solve the inequality $|2-5x| \leqslant 3$ using the three approaches.

X	Y1
-0.6	5
-0.4	4
-0.2	3
0	2
0.2	1
0.4	0
0.6	1
0.8	2
1	3
1.2	4
1.4	5

X=-0.2

Numerically: setting Y1 = abs(2−5X), we obtain the table shown in the figure with ΔTbl = 0.2. From this, we see that $|2-5x| \leqslant 3$ if $-0.2 \leqslant x \leqslant 1$.



window $[-2, 2, 1] \times [-2, 4, 1]$

Graphically: we obtain the graph shown in the figure. Finding the intersection points of the two lines, we see that $|2-5x| \leqslant 3$ if $-0.2 \leqslant x \leqslant 1$.

Algebraically

The inequality $|2-5x| \leqslant 3$ means

$$2-5x \leqslant 3 \quad \textbf{and} \quad -(2-5x) \leqslant 3.$$

Multiply the second inequality by -1 and therefore change the inequality direction to give

$$2-5x \leqslant 3 \quad \textbf{and} \quad 2-5x \geqslant -3,$$

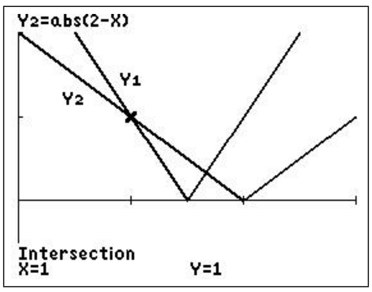
so that $-1 \leqslant 5x$ **and** $5 \geqslant 5x$. Dividing both by 5 and combining gives $-0.2 \leqslant x \leqslant 1$.

4. Solve the inequality $|2x-3| \leqslant |2-x|$ using the three approaches.

X	Y1	Y2
0	3	2
0.3333	2.3333	1.6667
0.6667	1.6667	1.3333
1	1	1
1.3333	0.3333	0.6667
1.6667	0.3333	0.3333
2	1	0
2.3333	1.6667	0.3333
2.6667	2.3333	0.6667
3	3	1
3.3333	3.6667	1.3333

X=1.66666666666

Numerically: with Y1 = abs(2X–3), Y2 = abs(2–X) and ΔTbl = 1/3, we obtain the table shown.
Then, $|2x-3| \leqslant |2-x|$ (Y1 ⩽ Y2) if $1 \leqslant x \leqslant \frac{5}{3}$.
►Frac (math 1) is useful here.



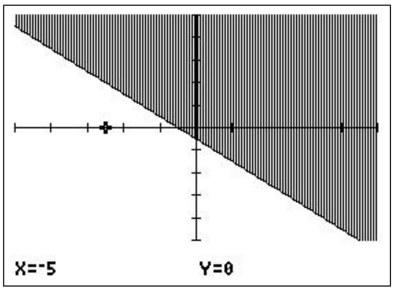
window $[0, 3, 1] \times [-0.5, 2, 1]$

Graphically: we obtain the graph shown. Finding the intersection points of the two graphs, we see that $|2x-3| \leqslant |2-x|$ (Y1 ⩽ Y2) if $1 \leqslant x \leqslant \frac{5}{3}$.

Algebraically: this gets messy. Better to use either of the other two methods.

5. Graph the region satisfying the inequality $1 < x + 2y$ for $-10 < x < 10$.

Rewrite as $y > \dfrac{-1-x}{2}$. Set Y1 = (−1−X)/2 (watch − signs) and shade above.



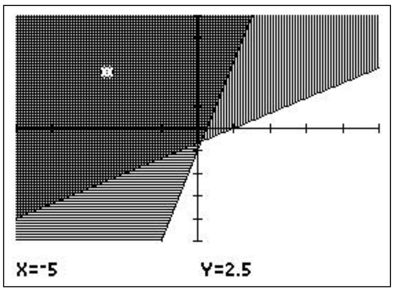
window $[-10, 10, 2] \times [-5, 5, 1]$

The point shown, $(-5, 0)$, does not satisfy the inequality, showing the shaded region does. Points on the boundary line $y = (-1-x)/2$ do not satisfy the inequality because of the $<$ sign.

6. Graph the region satisfying the inequalities $x - 3y \leqslant 2$ **and** $y \geqslant 2x - 1$.

Rewrite the first inequality as $\dfrac{x-2}{3} \leqslant y$.

Set Y1 = (X−2)/3, shade above, and Y1 = 2X−1, shade above.



window $[-10, 10, 2] \times [-5, 5, 1]$

The double-shaded region is the solution region. Check with a point in the region such as $(-5, 2.5)$.

Points on the boundary lines do satisfy the inequalities because of the $\leqslant$ signs.

4.9.2 Activities (Section 4.7.2)

The fruit basket

The produce manager of a grocery is making up fruit baskets to sell as gifts. They are to sell for no more than \$5, and contain only apples and oranges. She wants to get 24c per orange, 12c per apple and 68c for the basket. No more than 26 pieces of fruit will fit in the basket. Suppose she uses x oranges and y apples.

(a) Show that we are looking for those x and y values that satisfy the inequalities

$$\begin{array}{rlcrl} x & \geqslant 0 & \quad & x+y & \leqslant 26 \\ y & \geqslant 0 & \quad & 2x+y & \leqslant 36 \end{array}$$

The numbers of oranges and apples are zero or positive, so that

$$x \geqslant 0 \qquad y \geqslant 0.$$

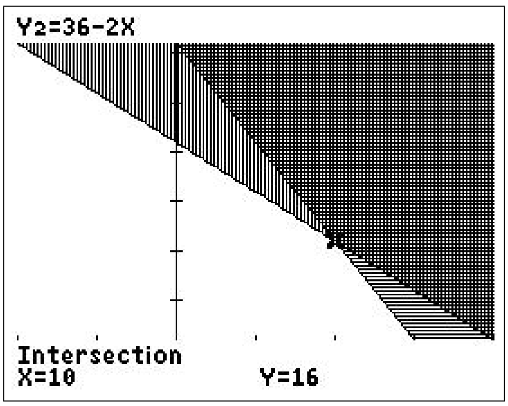
There are to be no more than 26 pieces of fruit, so that $x+y \leq 26$.

The total cost of fruit plus basket is to be no more than \$5, so that

$$\begin{aligned} 24x + 12y + 68 &\leqslant 500 \\ \text{or} \quad 2x + y &\leqslant 36. \end{aligned}$$

(b) When the equality signs are used we have the equations for four lines. Draw a diagram showing those four lines and then find an area of the xy plane in which useful x and y values must be found. Shetch that area.

$x=0$ and $y=0$ give the axes. Rewrite the other two inequalities as $y \leqslant 26-x$ and $y \leqslant 36-2x$ for graphing; set shaded above to give an unshaded solution region. The unshaded area bounded by the axes and the two lines contains the (x,y) values that satisfy all the inequalities. The two lines intersect at $(10,16)$.



window $[0,30,5]\times[0,30,5]$

(c) Which of these (x, y) values could the manager use?

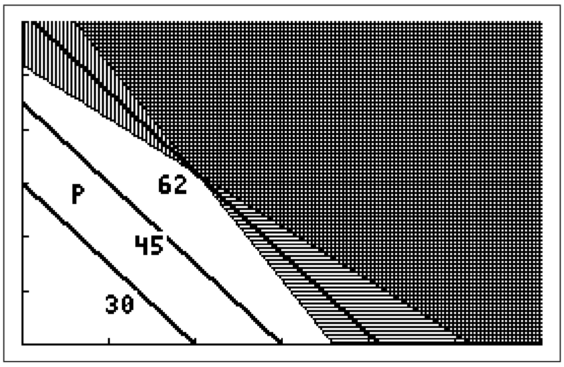
$$(5,10) \quad (10,5) \quad (5,25) \quad (20,5) \quad (-2,10)$$

$(5,10)$ and $(10,5)$ are in the unshaded area and so satisfy all the inequalities; these (x, y) values could be used. The other (x, y) values are in the shaded area and could not be used.

(d) If she makes a profit of 3 cents on every orange sold and 2 cents on every apple sold, what is the equation for the total profit P? Draw in the $P = 30$ and $P = 45$ lines on your diagram. Can you see how to get the maximum profit?

The profit is given by $P = 3x + 2y$, which we rewrite for graphing as $y = \frac{1}{2}(P - 3x)$.

The lines $P = 30$, $P = 45$ and $P = 62$ are shown in the figure below.



window $[0, 30, 5] \times [0, 30, 5]$

All profit lines will be parallel to those. The largest value of P is for the profit line farthest to the right and still intersecting the unshaded region, i.e. the line through the corner point (vertex) of the region $(x, y) = (10, 16)$, giving $P = 62$ from the profit equation.

The maximum profit is therefore 62 cents, obtained with 10 oranges and 16 apples in a basket.

Market garden

The area available for crops on a farm is $1000\,\text{m}^2$. Two crops grow well in the area and are being considered for planting. Each crop takes a different amount of time per m^2 to prepare the soil and plant. The total cost of this soil preparation and planting should not exceed \$2,500.

- Beans (per m^2): cost \$3; profit \$2.
- Spinach (per m^2): cost \$2; profit \$1.50.

What areas of each should be planted to maximise profit?

Let x (m^2) be the area of beans planted and y (m^2) the area of spinach. Then,

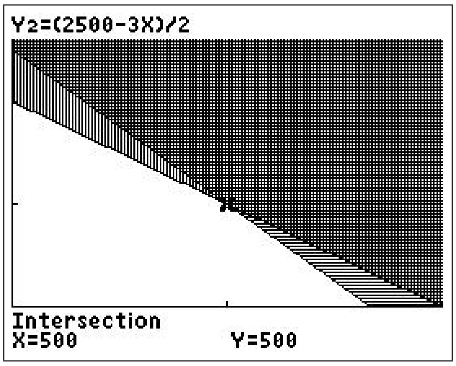
$x + y \leqslant 1000$ total area,

$3x + 2y \leqslant 2500$ total cost (\$),

$x,\, y \geqslant 0$ areas can't be negative;

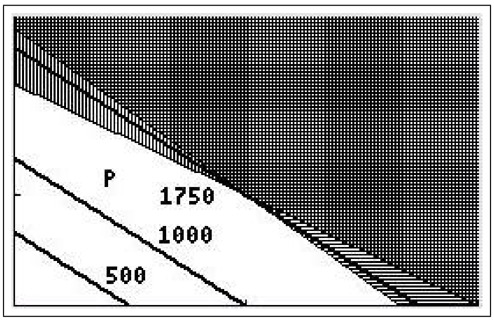
profit P(\$) is given by $P = 2x + 1.5y$.

Rewrite the first two inequalities as $y \leqslant 1000-x$ and $y \leqslant \frac{1}{2}(2500-3x)$; set shaded above to give an unshaded solution region. Graphing the four inequalities gives the figure below. The two curves intersect at the point $(500, 500)$.



window $[0, 1000, 500] \times [0, 1300, 500]$

Rewriting the profit equation gives $y = \frac{2}{3}(P-2x)$. Several profit curves are shown in the figure below. The profit curve with the greatest profit is the one that passes through the intersection point $(500, 500)$ (a vertex of the unshaded quadrilateral), giving $P = 1750$.



window $[0, 1000, 500] \times [0, 1300, 500]$

Therefore, $500\,\text{m}^2$ of each crop should be planted, giving a profit of \$1750.

Manufacturing #1

A factory manufactures doodads and whirligigs. It costs \$2 and takes 3 person hours to produce a doodad. It costs \$4 and takes 2 person hours to produce a whirligig. The factory has \$220 and 150 person hours a day to produce these products. If each doodad sells for \$6 and each whirligig sells for \$7, then how many of each product should be manufactured each day in order to maximise profit?

Let x be the number of doodads produced and y the number of whirligigs. Then,

$2x + 4y \leqslant 220$ total cost,

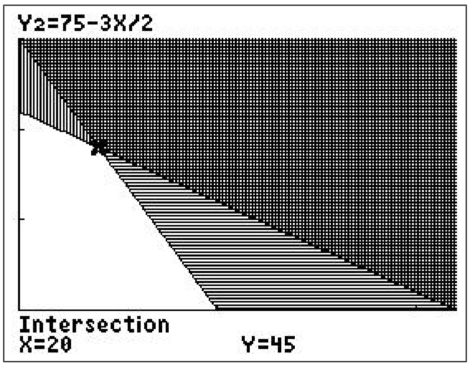
$3x + 2y \leqslant 150$ total hours,

$x,\, y \geqslant 0$ numbers produced can't be negative;

profit $P(\$)$ is given by $P = 4x + 3y$.

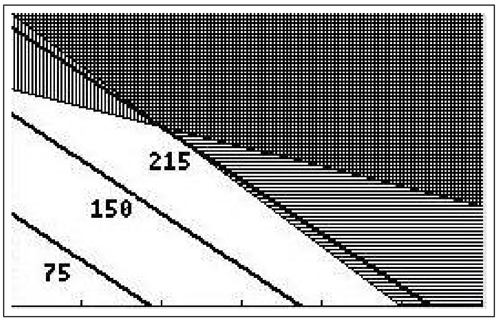
Rewrite the first two inequalities as $y \leqslant 55 - \dfrac{x}{2}$ and $y \leqslant 75 - \dfrac{3x}{2}$, and set shaded above to give an unshaded solution region.

Graphing the four inequalities gives the figure below. The two curves intersect at the point $(20, 45)$.



window $[0, 110, 50] \times [0, 75, 25]$

Rewriting the profit equation gives $y = \frac{1}{3}(P - 4x)$. Several profit curves are shown in the figure below: $P = 75$, $P = 150$ and $P = 215$. The profit curve with the greatest profit is the one that passes through the intersection point $(20, 45)$ (a vertex of the unshaded quadrilateral), giving $P = 215$.



window $[0, 60, 10] \times [0, 75, 25]$

Therefore, 20 doodads and 45 whirligigs should be produced each day, giving a profit of \$215.

Manufacturing #2

A manufacturer has 750 m of cotton fabric and 1000 m of polyester fabric. Production of a sweatshirt requires 1 m of the cotton and 2 m of the polyester, while production of a shirt requires 1.5 m of the cotton and 1 m of the polyester. The sale prices of a sweatshirt and a shirt are \$30 and \$24, respectively. How many of each type of shirt should be produced to maximise the return on sales (assuming all are sold)?

Let x be the number of sweatshirts produced and y the number of shirts. Then,

$x + 1.5y \leqslant 750$ amount of cotton used,

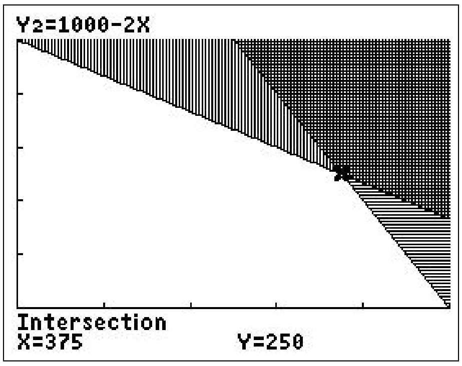
$2x + y \leqslant 1000$ amount of polyester used,

$x,\, y \geqslant 0$ numbers produced can't be negative;

the return on sales $R(\$)$ is $R = 30x + 24y$.

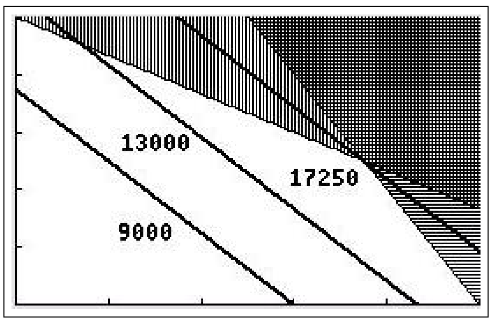
Rewrite the first two inequalities as $y \leqslant \frac{2}{3}(750-x)$ and $y \leqslant 1000-2x$, and set shaded above to give an unshaded solution region.

Graphing the four inequalities gives the figure below. The two curves intersect at the point $(375, 250)$.



window $[0, 500, 100] \times [0, 500, 100]$

Rewriting the return equation gives $y = \frac{1}{24}(R-30x)$. Several return curves are shown in the figure below, starting at the bottom left: $R = 9000$; $R = 13,000$; and $R = 17,250$. The curve with the greatest return is the one that passes through the intersection point $(375, 250)$ (a vertex of the unshaded quadrilateral), giving $R = 17,250$.



window $[0, 500, 100] \times [0, 500, 100]$

Therefore, 375 sweatshirts and 250 shirts should be produced, giving a total return of \$17,250.

Brainbuilding

A student is looking for supplement protein bars to help his brain work faster, and there are two available products: protein bar A and protein bar B.

Each protein bar A contains 15 g of protein and 30 g of carbohydrate, and has a total of 200 calories. Each protein bar B contains 30 g of protein and 20 g of carbohydrate, and has a total of 240 calories.

According to his nutritional plan, this student needs at least 20, 000 calories from these supplements over the month, which must comprise at least 1800 g of protein and at least 2200 g of carbohydrates.

If a protein bar A costs \$3 and a protein bar B \$4, what is the least possible amount of money he can spend to meet all his one-month requirements?

Assume he buys x of protein bar A and y of protein bar B in a month. Then,

$200x + 240y \geqslant 20,000$ total calories,

$15x + 30y \geqslant 1800$ total protein (g),

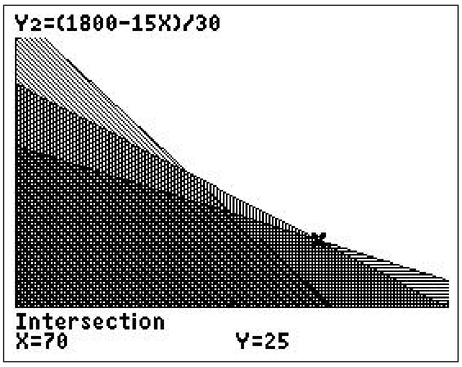
$30x + 20y \geqslant 2200$ total carbohydrate (g),

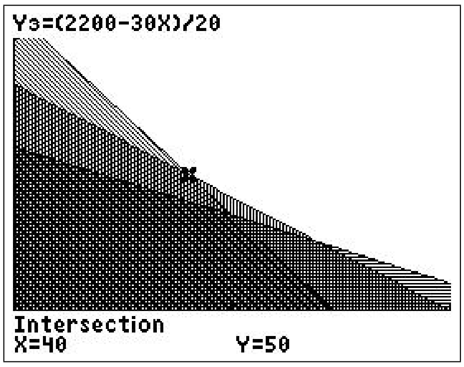
$x,\ y \geqslant 0$ quantities must be non-negative;

the total cost $C(\$)$ is given by $C = 3x + 4y$ (\$).

Rewrite the first 3 inequalities as $y_1 \geqslant \frac{1}{240}(20,000 - 200x)$, $y_2 \geqslant \frac{1}{30}(1800 - 15x)$ and $y_3 \geqslant \frac{1}{20}(2200 - 30x)$. Set shaded below to give an unshaded solution region.

Graphing these gives the figure below. Two intersection points of the three curves are vertices of the unshaded region; one of these must give the solution. y_1 and y_2 intersect at the point $(70, 25)$, y_1 and y_3 at the point $(40, 50)$.





window $[0, 100, 50] \times [0, 100, 50]$

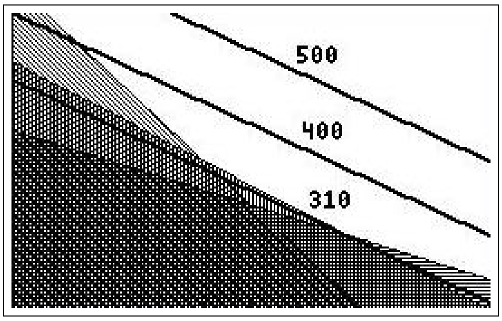
At this stage, knowing that one of these points must provide the solution, we could simply substitute both points into the cost equation to see which gives the lower cost.

The point $(40, 50)$ gives $C = 3 \times 40 + 4 \times 50 = 320$.

The point $(70, 25)$ gives $C = 3 \times 70 + 4 \times 25 = 310$, which is therefore the solution.

However, plotting a few cost curves shows us why this is the solution.

Rewriting the cost equation gives $y=\frac{1}{4}(C-3x)$. Several cost curves are shown in the figure below: $C=500$; $C=400$; and $C=310$. The curve that passes through the intersection point $(70,25)$ has $C=310$, that through the intersection point $(40,50)$ has $C=320$.



window $[0,100,50]\times[0,100,50]$

Therefore, he should buy, monthly, 70 of protein bar A and 25 of protein bar B, at a total cost of \$310.

Feeding the cows

A farmer feeds his cows a feed mix to supplement their foraging. The farmer uses two types of feed for the mix. Corn feed contains 100 g of protein per kg and 750 g of starch per kg. Wheat feed contains 150 g of protein per kg and 700 g of starch per kg. Each cow should be fed at most 7 kg of feed per day. The farmer would like each cow to receive at least 650 g of protein and 4000 g of starch per day. If corn feed costs \$0.40/kg and wheat costs \$0.45/kg, what is the feed mix that minimises cost? Round your answers to the nearest gram.

Assume he feeds each cow x kg of corn feed and y kg of wheat feed each day. Then, noting the different inequalities,

$x+y\leqslant 7$ total amount fed per cow (kg),

$100x+150y\geqslant 650$ total protein (g),

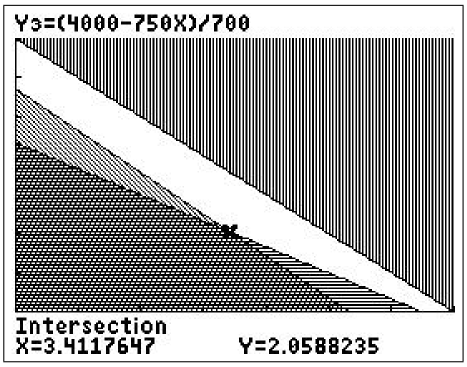
$750x+700y\geqslant 4000$ total starch (g),

$x,\, y\geqslant 0$ quantities must be non-zero;

the total cost $C(\$)$ is given by $C=0.4x+0.45y$.

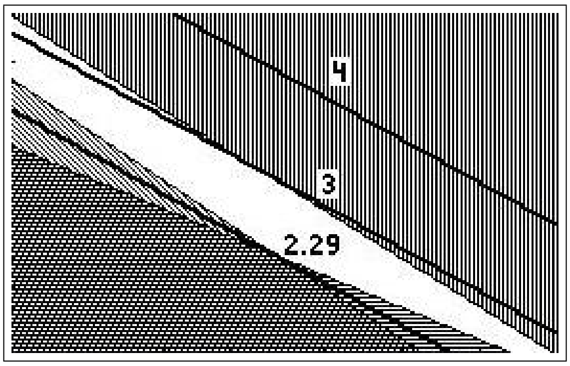
Rewrite the first 3 inequalities as $y\leqslant 7-x$, $y\geqslant\frac{1}{150}(650-100x)$ and $y\geqslant\frac{1}{700}(4000-750x)$.

On graphing the inequalities, shaded appropriately to give an unshaded solution region, we obtain the figure here.
The bottom two curves intersect at the point $(3.412, 2.059)$ which, according to the theory, must be the solution because it is the only vertex (intersection) in the region.



window $[0,7,1]\times[0,7,1]$

Checking graphically, rewriting the cost equation gives $y=\frac{1}{0.45}(C-0.4x)$. Several cost curves are shown in the figure below, starting at the top right: $C=4$; $C=3$; and $C=2.29$. Clearly, the curve with the least cost is the one that passes through the intersection point of the bottom two curves $(3.412, 2.059)$ (a vertex of the unshaded quadrilateral), giving $C=2.29$.



window $[0, 7, 1] \times [0, 7, 1]$

Therefore, the farmer should feed each cow 3.412 kg of corn feed and 2.059 kg of wheat feed each day (rounded to the nearest gram), at a total cost of $2.29 per cow.

Pizza profits

A company produces two frozen pizzas, the Gluttono and the Carnivore. The three main ingredients used in both pizzas are cheese, tomato and meat. The quantity of these ingredients required for each pizza and the weekly supply of these are listed in the table below.

Ingredient	**Grams required per Gluttono**	**Grams required per Carnivore**	**Weekly supply** (kg)
cheese	70	80	96
tomato	60	40	70
meat	10	60	48

The company makes a profit of $4 on each Gluttono and $5.50 on each Carnivore.

(a) How many of each type of pizza should the company produce each week to maximise the profit?

Let x and y be the numbers of Gluttono and Carnivore pizzas produced per week. Then,

$70x + 80y \leqslant 96,000$ quantity of cheese (g),

$60x + 40y \leqslant 70,000$ quantity of tomato (g),

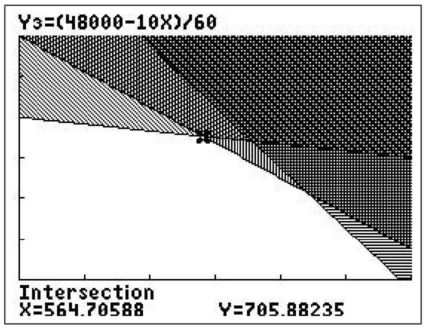
$10x + 60y \leqslant 48,000$ quantity of meat (g),

$x,\ y \geqslant 0$ quantities must be non-zero;

the total profit $P(\$)$ is given by $P = 4x + 5.5y$.

Rewrite the inequalities as $y \leqslant \frac{1}{80}(96000–70x)$, $y \leqslant \frac{1}{40}(70000–60x)$ and $y \leqslant \frac{1}{60}(48000–10x)$.

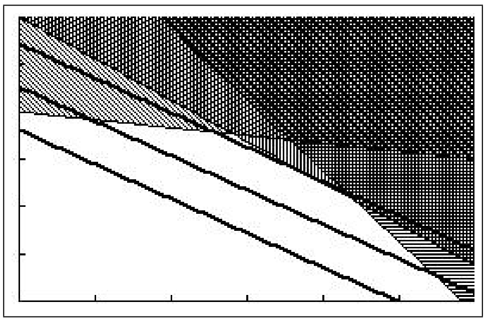
On graphing the inequalities, shaded appropriately to give an unshaded solution region, we obtain the figure here.
Y1 and Y3 intersect at the point $(564.7, 705.9)$ (shown) and Y1 and Y2 intersect at the point $(880, 430)$.



window $[0, 1200, 200] \times [0, 1200, 200]$

Rewriting the profit equation gives $y = \frac{1}{5.5}(P - 4x)$. Several cost curves are shown in the figure below, starting at the bottom left: $P = 4000$; $P = 5000$; and $C = 6000$.

Clearly, the curve with the greatest profit is the one that passes through the intersection point of Y1 and Y3, $(564.7, 705.9)$ (a vertex of the unshaded quadrilateral), giving $P = 6141$.



window $[0, 1200, 200] \times [0, 1200, 200]$

Therefore, the maximum profit of \$6141 is generated if 564.7 Gluttonos and 705.9 Carnivores are produced each week.

These numbers should be rounded down to 564 and 705 to be appropriate to the problem and still fit the constraints. The maximum profit is then \$6133.50.

(b) What quantity of ingredients, if any, are left unused when the maximum profit is generated?

Substituting $x = 564$ and $y = 705$ back into the original constraint equations gives the amount of cheese used as 95.88 kg, so that only 120 g of cheese is unused at the end of the week. Similarly, 8 kg of tomatoes and 60 g of meat are unused.

This seems cutting it a bit fine for cheese and meat, so maybe some adjustment is needed.

Extensions

(c) If the amount of cheese available increases to 110 kg per week, what is the effect on production, given the company still strives to maximise its profit?

Cheese no longer comes into consideration. Only one intersection point (Y2 and Y3) at $(712.5, 681.25)$, giving 712 Gluttonos and 681 Carnivores and a profit of \$6593.50.

(d) What is the effect of a 10% decrease in the profit made on a Gluttono pizza?

Same optimum point but profit now \$5907.90.

(e) What change in profit on the Gluttono pizza would alter the optimal solution?

For the other intersection point (Y1 and Y2) to be the optimal solution, we need the profit-line slope $-P_G/P_C$ steeper than that of Y1, which is $-7/8$. Therefore, with $P_C = 5.5$, need $P_G > 4.8125$.

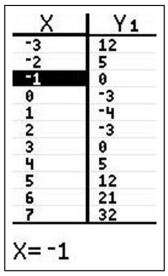
The profit on the Gluttono pizza would have to be greater than \$4.82 to alter the optimal solution.

4.9.3 Quadratic inequalities (Section 4.8.2)

Solve the following inequalities using any or all of the three approaches.

1. $x^2 - 2x - 3 \leqslant 0$.

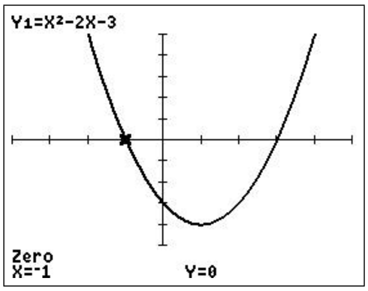
Numerically: setting $\text{Y}_1 = \text{X}^2 - 2\text{X} - 3$ and $\Delta\text{Tbl} = 1$, we obtain the table shown in the figure.
Then, $x^2 - 2x - 3 \leqslant 0$ if $-1 \leqslant x \leqslant 3$.



X	Y1
-3	12
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5
5	12
6	21
7	32

X=-1

Graphically: setting $\text{Y}_1 = \text{X}^2 - 2\text{X} - 3$, we obtain the graph shown in the figure. Finding the zeros (calc menu), we see that $x^2 - 2x - 3 \leqslant 0$ if $-1 \leqslant x \leqslant 3$.



window $[-4, 5, 1] \times [-5, 5, 1]$

Algebraically

To solve the inequality, we then need to factorise the quadratic:

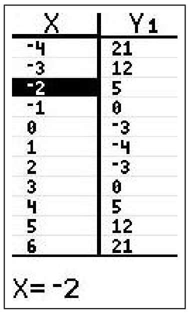
$x^2 - 2x - 3 = (x+1)(x-3)$.

$$\text{Then, } (x+1)(x-3) \leqslant 0 \implies \begin{cases} x+1 \leqslant 0 \textbf{ and } x-3 \geqslant 0 & (1) \\ \textbf{or} & \\ x+1 \geqslant 0 \textbf{ and } x-3 \leqslant 0. & (2) \end{cases}$$

Equation (1) gives $x \leqslant -1$ **and** $x \geqslant 3$, which has no solution.

Equation (2) gives $x \geqslant -1$ **and** $x \leqslant 3$, that is $-1 \leqslant x \leqslant 3$, which is the algebraic solution.

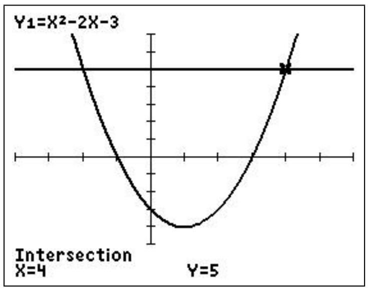
2. $x^2 - 2x - 3 \leqslant 5$.



X	Y1
-4	21
-3	12
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5
5	12
6	21

X= -2

Numerically: setting $Y_1 = X^2 - 2X - 3$, $\Delta\text{Tbl} = 1$, we obtain the table shown in the figure. From this, $x^2 - 2x - 3 \leqslant 5$ if $-2 \leqslant x \leqslant 4$.



window $[-4, 6, 1] \times [-5, 7, 1]$

Graphically: setting $Y_1 = X^2 - 2X - 3$ and $Y_2 = 5$, we obtain the graph shown in the figure. Finding the zeros (calc menu), we see that $x^2 - 2x - 3 \leqslant 5$ if $-2 \leqslant x \leqslant 4$.

Algebraically

First, rewrite the inequality as $x^2 - 2x - 8 \leqslant 0$.

To solve the inequality, we then need to factorise the quadratic:

$x^2 - 2x - 8 = (x+2)(x-4)$.

Then, $(x+2)(x-4) \leqslant 0 \quad \Longrightarrow \quad x+2 \leqslant 0$ **and** $x-4 \geqslant 0 \qquad (1)$

or

$x+2 \geqslant 0$ **and** $x-4 \leqslant 0. \qquad (2)$

Equation (1) gives $x \leqslant -2$ **and** $x \geqslant 4$, which has no solution.

Equation (2) gives $x \geqslant -2$ **and** $x \leqslant 4$, that is $-2 \leqslant x \leqslant 4$, which is the algebraic solution.

PTO

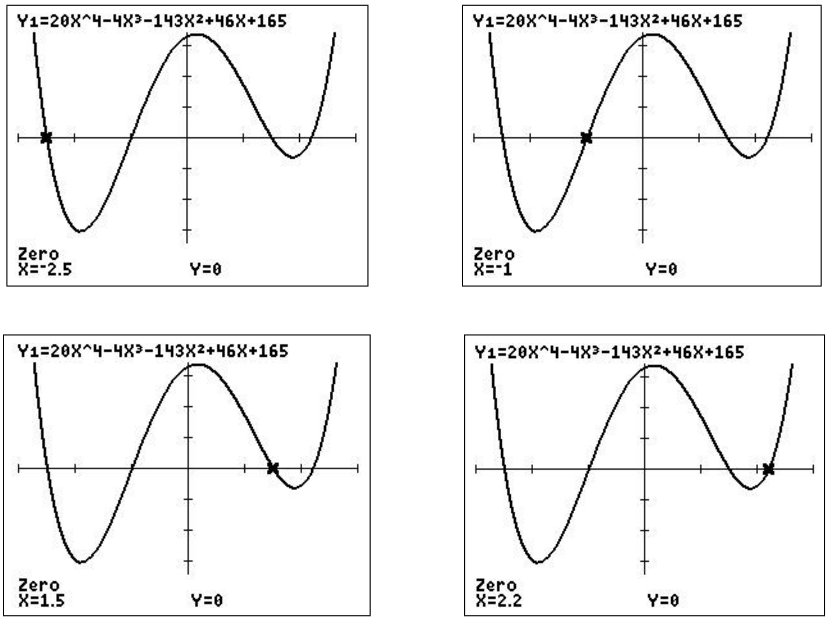
3. $20x^4 - 4x^3 - 143x^2 + 46x + 165 < 0.$

The only sensible way to do this is graphically.

Set $Y_1 = 20X^4 - 4X^3 - 143X^2 + 46X + 165$ and graph with an appropriate window.

As this is a fourth-degree polynomial, we expect up to four zeros.

Using [calc] zero:



window $[-3, 3, 1] \times [-170, 170, 50]$

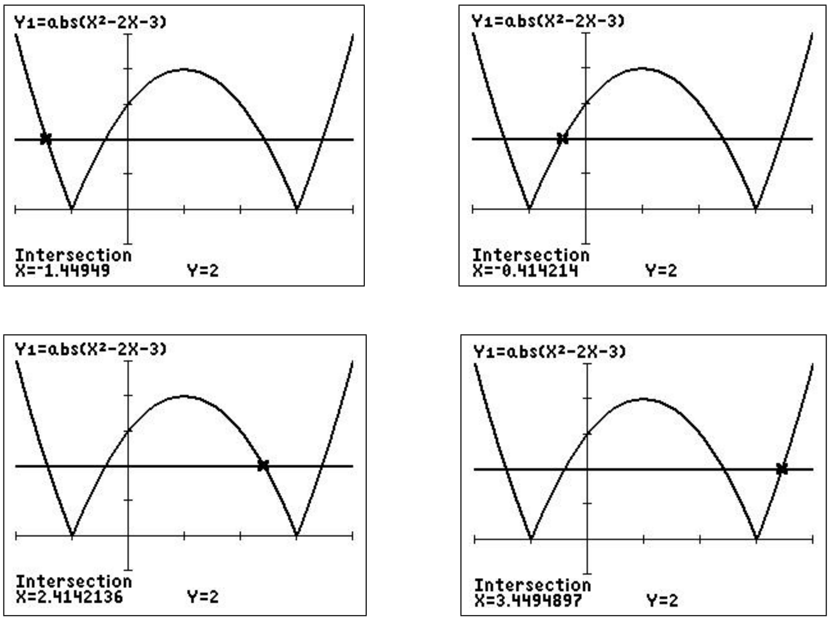
From the graphs, $20x^4 - 4x^3 - 143x^2 + 46x + 165 < 0$ if $-2.5 < x < 1$ or $1.5 < x < 2.2$.

PTO

4. $|x^2-2x-3| \leqslant 2$.

Again, the only sensible way to do this is graphically. Set $Y_1 = \text{abs}(X^2-2X-3)$, $Y_2=2$ and graph with an appropriate window.

Using calc intersect:



window $[-2,4,1]\times[-1,5,1]$

Therefore, $|x^2-2x-3| \leqslant 2$ if $-1.449 \leqslant x \leqslant -0.4142$ or $2.414 \leqslant x \leqslant 3.449$, approximately.

5 Fitting Curves to Data 1: Calculator Functions

5.1 Introduction

Much scientific and other research involves data. Fitting a function to the data is a way of summarising the data; if the fit is good, the fitted curve can be used instead of the data in further calculations, especially useful if Calculus is involved. Sometimes the function chosen is guided by the theory involved, for example motion under gravity. At other times, the choice is empirical: the function which gives the best fit is used.

The material here is especially relevant to the second case. The functions available on the TI-84/CE are given here, together with an example of each in fitting a given dataset.

Fitting Curves to Data 2 is in the third volume of this book. It gives a matrix method to fit polynomials exactly to data, goes through the 19 curves available for fitting in the CURVEFIT/CRVFITCE program and looks at time averaging of data.

5.1.1 Setting up

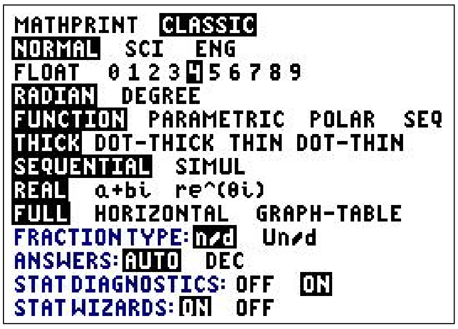
The screens below show the TI-84CE mode (below left) and format (below right) settings used here; the TI-84Plus MODE and FORMAT screens contain the same settings that are required for our purposes here.

Two mode settings are directly relevant.

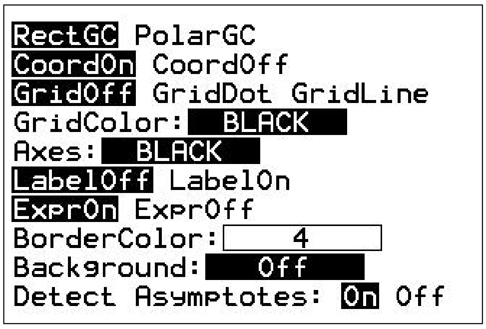
STAT DIAGNOSTICS ON means that the goodness-of-fit parameters r and r^2 or R^2 are displayed.

STAT WIZARDS ON means a menu appears for the inputs to a command, rather than you having to specify them with the command (as in the commands in Section 5.2 below).

An alternative is to scroll down to a command and press +; a Catalog Help[19] screen appears showing the commands required and in what order. You can then type them in on this screen and paste the whole command onto the Home screen for execution.



mode screen



format screen

In CLASSIC mode, commands are typed on one line, with arguments in brackets. In MATHPRINT mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for inputs. This applies to the powers and roots (blue) commands on the keyboard and to many of the commands in the math NUM menu. Either setting is fine here.

[19] You may have to download the Catalog Help app for an 84Plus from *education.ti.com*; it is built into the 84CE operating system.

5.2 Operations on the calculator

The curve-fitting commands are in the stat CALC menu (screens below), starting at 3:Med-Med.

```
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7:QuartReg
8:LinReg(a+bx)
9↓LnReg
```

```
EDIT CALC TESTS
7↑QuartReg
8:LinReg(a+bx)
9:LnReg
0:ExpReg
A:PwrReg
B:Logistic
C:SinReg
D:Manual-Fit Y=mX+b
E:QuickPlot&Fit-EQ
```

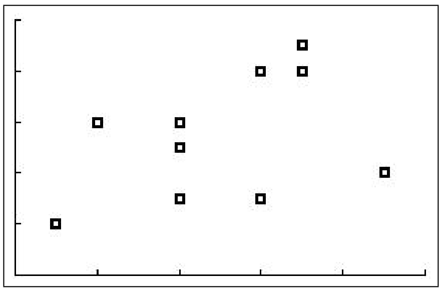
The data shown below left and graphed below centre are used for all the fits here. The *Stats Wizard* input menu for most of the operations is the same except for the title at the top: that for the *Med-Med* command is shown below right.[20] If the regression equation, RegEQ, is specified on this screen, the fitted curve will be drawn over the data when you press graph.

Y1 is in the vars Y-VARS Function menu or via the shortcut alpha trace (F4).

To execute the command, move the cursor to *Calculate* and press enter.

L1	L2
1	2
2	6
4	5
4	6
4	3
6	3
6	8
7	9
7	8
9	4

data



window $[0, 10, 2] \times [0, 10, 2]$

```
Med-Med
Xlist:L1
Ylist:L2
FreqList:
Store RegEQ:Y1
Calculate
```

input menu

Alternatively, scroll down the stat CALC menu to *Med-Med*, press + to go the CATALOG HELP page for this operation and fill in the the required inputs (screen below). Defaults are L1, L2 and no equation, i.e. just the command. Press PASTE to paste the full command onto the Home screen and enter to execute it.

```
CATALOG HELP
Med-Med L1,L2,Y1

[Xlistname,Ylistname
 ,freqlist,regequ]

PASTE ESC
```

Note: Values here for the coefficients in the equations, and the parameters r, r^2 and R^2 have been rounded to 4 decimal places (on the mode screen).

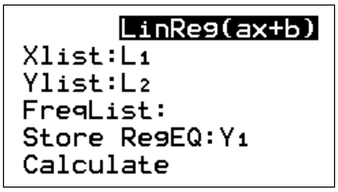
[20]This only difference is in fitting sine curves (Section 5.2.9), for which the number of iterations (default 3) and an optional period can also be specified.

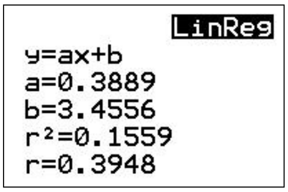
5.2.1 Linear functions

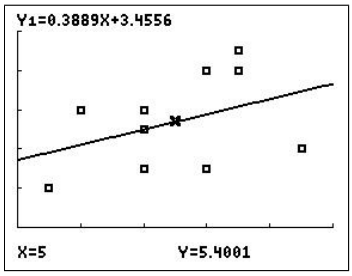
Linear regression, the fitting of straight lines to data, is the workhorse of simple data fitting. Provided the regression coefficient r is not exactly zero, all sorts of claims are made that the data are linear. The other methods presented here allow you to go beyond linear regression easily. If some other type of function gives a much better fit, you can start thinking whether there might be a good reason for this.

LinReg(ax+b) (linear regression) fits the model equation $y = ax + b$ to the data using a least-squares fit. It displays values for a (slope) and b (y-intercept); it also displays values for r^2 and r (when *DiagnosticOn* is set).

With STAT WIZARDS ON ,[21] selecting *LinReg*$(ax+b)$ gives the screen below left (after filling it in), executing it (cursor on *Calculate* and press enter) gives the screen below centre and pressing graph gives the screen below right.
Y1 is in the vars Y-VARS Function... menu or press alpha trace 1 .





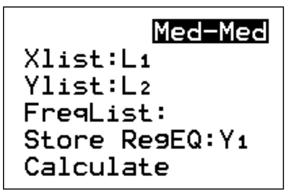


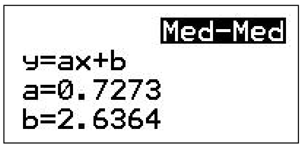
window $[0, 10, 2] \times [0, 10, 2]$

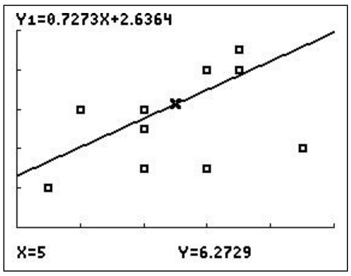
LinReg$(a+bx)$ (linear regression) similarly fits the model equation $y = a + bx$ to the data using a least-squares fit. The output is the same as for *LinReg*$(ax+b)$ above but with a and b interchanged.

The least-squares regression line above is relatively easy to calculate and has a sound theoretical foundation to justify its use, but outliers can have a large effect on the line. The median-median regression line is a more-resistant alternative. To obtain the line, the data points are divided into three equal groups by size of x.[22] For the low and high groups, the medians of the x and y values are obtained separately, giving one summary point for each group. The median-median regression line is the line joining these two points.

Med-Med (median-median) fits the model equation $y = ax + b$ to the data using the median-median regression technique and displays values for a (slope) and b (y-intercept).







window $[0, 10, 2] \times [0, 10, 2]$

[21] Regardless of whether you specify a *RegEQ*, the regression equation is always stored to the variable *RegEQ* in the vars Statistics... EQ menu.

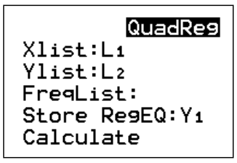
[22] If there is one extra point, the middle group is made one larger; if there are two extra points, the low and high groups are each made one larger.

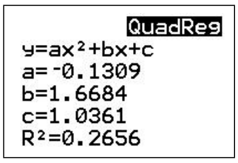
Manual-Fit Y=mX+b allows you to drop points on the screen and visually fit a linear function. See the TI-84 manual for details.

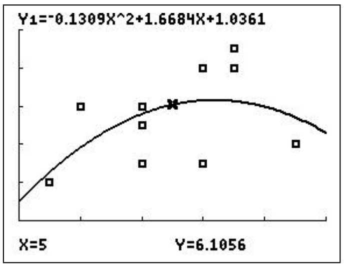
5.2.2 Quadratic functions

Quadratic functions occur in many areas involving data, with perhaps the best-known being the position of a body moving vertically under gravity. As in this case, theory often directs you to using a quadratic fit. Any set of data with a single (local) maximum or minimum is a candidate for a quaratic fit.

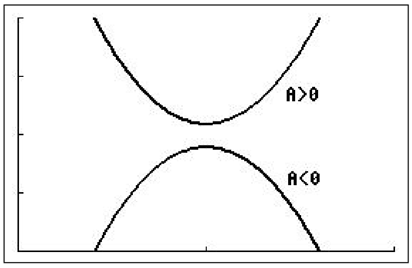
QuadReg (quadratic regression) fits the second-degree polynomial $y = ax^2 + bx + c$ to the data. It displays values for a, b and c, and R^2. For three data points, the equation is a polynomial fit; for four or more, it is a polynomial regression. At least three data points are required.







window $[0,10,2]\times[0,10,2]$



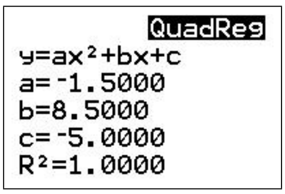
general shape of quadratics

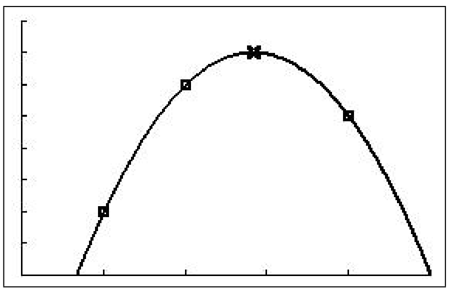
Finding maxima or minima of data

A quick way of finding (an approximation to) the maximum (minimum) of a set of data is to fit a quadratic function to the top (bottom) three points (and store the quadratic in Y_1 if you want to calculate the y value of the maximum/minimum). The maximum (minimum) of the data is then given approximately by $x_m = -b/(2a)$ and $y_m = Y_1(x_m)$.

This works best if the highest (lowest) point is the middle point.

Example: Consider the first three points in the data used so far. A (perfect) quadratic fit to these points gives the result below left. Immediately, $x_m = -8.5/(2\times -1.5) = 2.83$ and $y_m = Y_1(x_m) = 7.04$. Graphing the fitted parabola shows this maximum point (below right).





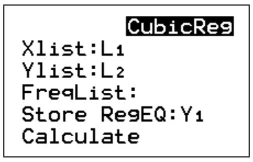
window $[0,5,1]\times[0,8,1]$

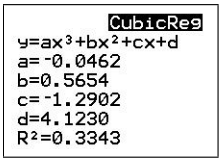
Of course, once you have plotted the parabola, you can also use *maximum* in the calc menu to find the maximum point.

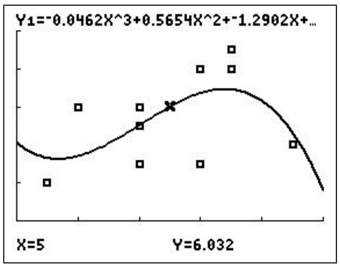
5.2.3 Cubic functions

Less common than quadratics, there are nevertheless theories that predict cubic behaviour. Any set of data with a (local) minimum and (local) maximum is a candidate for a cubic fit.

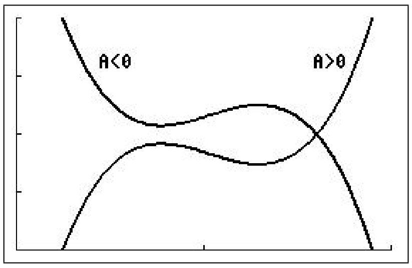
CubicReg (cubic regression) fits the third-degree polynomial $y = ax^3 + bx^2 + cx + d$ to the data. It displays values for a, b, c and d, and R^2. For four points, the equation is a polynomial fit; for five or more, it is a polynomial regression. At least four points are required.







window $[0, 10, 2] \times [0, 10, 2]$

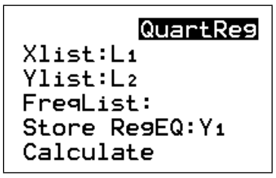


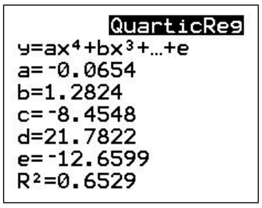
general shape of cubics

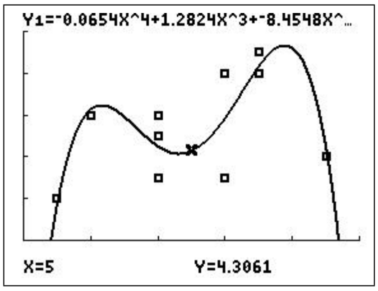
5.2.4 Quartic functions

Quartic functions, in general, are characterised by either one (local) minimum and two (local) maxima (negative a; as in the figure below) or vice versa (positive a).

QuartReg (quartic regression) fits the fourth-degree polynomial $y = ax^4 + bx^3 + cx^2 + dx + e$ to the data. It displays values for a, b, c, d and e, and R^2. For five points, the equation is a polynomial fit; for six or more, it is a polynomial regression. At least five points are required.





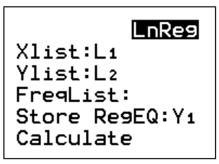


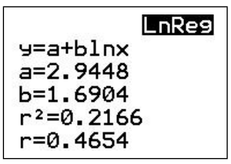
window $[0, 10, 2] \times [0, 10, 2]$

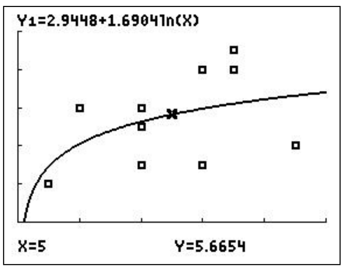
5.2.5 Logarithmic functions

Logarithmic functions $y = a + b\ln(x)$ with $b < 0$ appear to approach some finite value asymptotically but actually increase without bound; those with $b > 0$ tend asymptotically but slowly to zero.

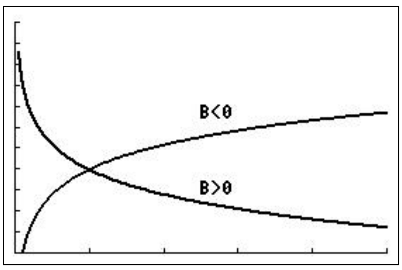
LnReg (logarithmic regression) fits the equation $y = a + b\ln(x)$ to the data using a least-squares fit. It displays values for a and b, and r^2 and r. The fit is done by turning the logarithmic function into a linear function by defining a new independent variable $X = \ln(x)$.







window $[0, 10, 2] \times [0, 10, 2]$



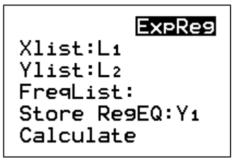
general shape of log curves

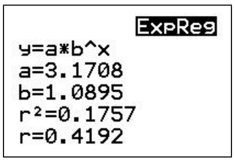
5.2.6 Exponential functions

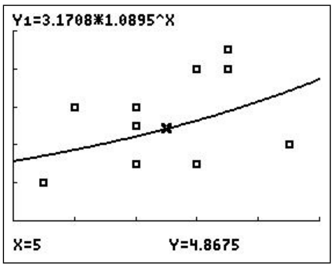
Exponential functions are common in mathematical modelling and associated data; the most common form is *the* exponential e^{ax}. With $a<0$, it models decay (for example, radioactive decay or declining populations), absorption (for example, of light) and slowing down (for example, air resistance and friction). Positive a is less common, as this represents exponential growth, which is unsustainable over any distance or time.

ExpReg (exponential regression) fits the model equation $y = ab^x$ to the data using a least-squares fit. It displays values for a and b, and r^2 and r. The fit is done by taking natural logs of both sides of the equation, giving $\ln(y) = \ln(a) + x\ln(b)$, then turning this into a linear function by defining a new dependent variable $Y = \ln(y)$.

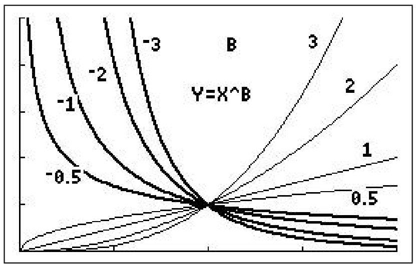
The right-hand figure shows several exponential functions with positive and negative exponents to demonstrate the general shape.







window $[0,10,2]\times[0,10,2]$

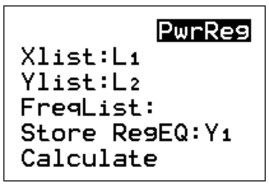


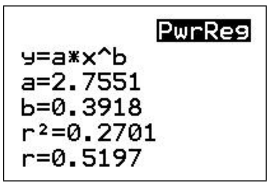
window $[0,2,0.25]\times[0,5,1]$

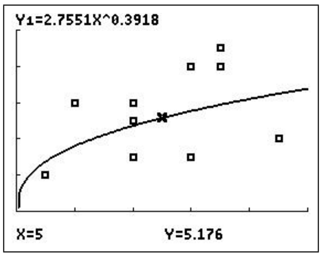
5.2.7 Power functions

PwrReg (power regression) fits the model equation $y = ax^b$ to the data using a least-squares fit with transformed values $\ln(x)$ and $\ln(y)$. It displays values for a and b, and r^2 and r. The fit is done by taking natural logs of both sides of the equation, giving $\ln(y) = \ln(a) + b\ln(x)$, then turning this into a linear function by defining new variables $Y = \ln(y)$ and $X = \ln(x)$. Log-log plots are often used to display data with a large range of values, particularly in Biology. A log-log plot turns a power function into a straight line of slope b, the transformation here.

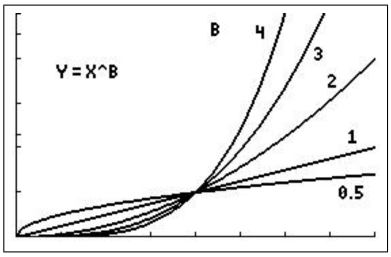
The right-hand figure shows several power functions to demonstrate the general shape.







window $[0,10,2]\times[0,10,2]$

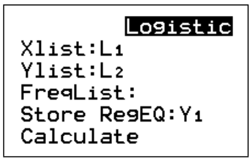


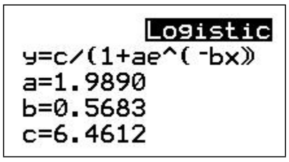
window $[0,2,1]\times[0,5,1]$

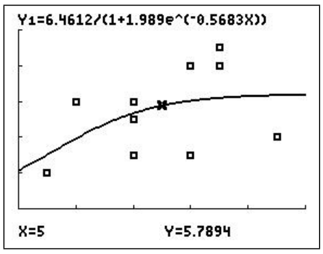
5.2.8 Logistic functions

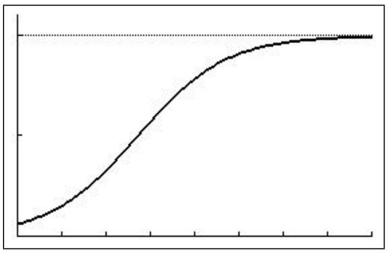
Logistic functions were one of the early attempts at a population model that was more realistic than exponential growth. The growth rate in a logistic population model decreases as the population increases, with the result that, after an initial almost exponential growth, the population tends asymptotically to a finite value called the carrying capacity.

Logistic fits the model equation $y = c/(1+ae^{-bx})$ to the data using an iterative least-squares fit. It displays values for a, b and c. The right-hand figure shows the general shape of a logistic function, tending asymptotically to the horizontal dotted line $y = c$ (assuming $b > 0$).







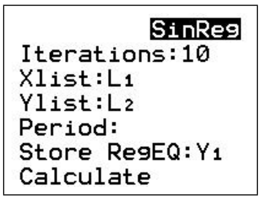


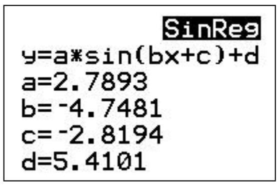
window $[0, 10, 2] \times [0, 10, 2]$ general logistic-curve shape

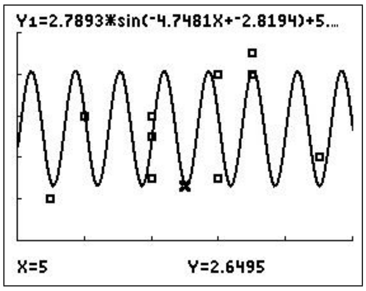
5.2.9 Sine functions

Sine (and cosine) functions, along with exponentials, are the most common functions in modelling; they model oscillations. Together, sines and exponentials model damped oscillations, such as the motion of a pendulum, and form the basis for all models of oscillating and vibrating systems.

SinReg (sinusoidal regression) fits the model equation $y = a\sin(bx+c) + d$ to the data using an iterative least-squares fit. It displays values for a, b, c and d. At least four data points are required. At least two data points per cycle are required in order to avoid aliased frequency estimates.







window $[0, 10, 2] \times [0, 10, 2]$

Iterations (above left) is the maximum number of times the algorithm will iterate to find a solution. The value for *Iterations* can be an integer $\geqslant 1$ and $\leqslant 16$; if not specified, the default is 3. The algorithm may find a solution before *Iterations* is reached. Typically, larger values for *Iterations* result in better accuracy for SinReg but longer execution times.

A value for *Period* is optional. If you do not specify *Period*, the difference between time values in *Xlist* must be equal and the time values must be ordered in ascending sequential order.[23] If you specify *Period*, the algorithm may find a solution more quickly, or it may find a solution when it would not have found one if you had omitted a value for *Period*. If you specify *Period*, the differences between time values in *Xlist* can be unequal.

[23]This was not the case for the data here, but it still seemed to obtain a reasonable fit.

With noisy data, you will achieve better convergence results when you specify an accurate estimate for *Period*. You can obtain a value in either of two ways.

- Plot the data and trace to determine the x distance between the beginning and end of one complete period, or cycle.
- Plot the data and trace to determine the x distance between the beginning and end of N complete periods, or cycles. Then divide the total distance by N.

After your first attempt to use SinReg with the default value for *Iterations*, you may find the fit to be approximately correct, but not optimal. For an optimal fit, execute the SinReg command again with $\mathit{Iterations} = 16$ and $\mathit{Period} = 2\pi/b$, where b is the value obtained from the previous execution.

5.2.10 Manual-Fit

Fits a linear equation to a scatter plot and allows you to change the fit parameters on screen. See the TI-84 manual for details.

5.2.11 Quick Fit (CE)

QuickPlot & Fit-EQ allows you to drop points on a graph screen and fit a curve to those points using regression functions. You can select colour and line style, draw points on a graph and choose an equation to fit the points. You can then store the results of the plot and the equation. See the TI-84 manual for details.

6 Population Modelling 1: Exponential Growth

6.1 Introduction to population modelling

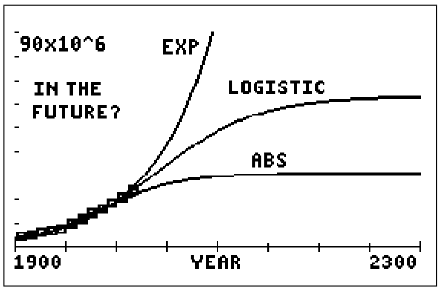
When mathematicians talk about playing with a model, chances are they don't mean a model plane or boat. They are probably talking about a mathematical model — a set of equations that describe in mathematics how a particular system works. There are mathematical models for many things, such as the planets revolving about the Sun, heating iron ore in a blast furnace, pollution in a lake, how prices vary on the stock exchange, the spread of diseases and how populations (people, animals, bacteria, viruses, etc) change with time.

Population modelling started a long time ago, and one of the earliest modellers was Fibonacci (1170 – 1250). In his book *Liber abaci*, he modelled a rabbit population, starting with one pair of baby rabbits. If each adult pair of rabbits produces only one pair of baby rabbits each month, and if baby rabbits take one month to become adults, the numbers of pairs of rabbits in successive months are given by the famous Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, and so on. The next number is found by adding the previous two numbers. Fibonacci numbers are also found elsewhere in Nature. If you look at a pine cone, you will find the 'petals' spiral in two directions. The number of petals it takes to get once around is almost always a Fibonacci number. The same thing occurs in pineapples, sunflowers and many other flowers.

Much later, Thomas Malthus (1766 – 1834) in England startled the world by predicting that food would run out sometime in the future because of the rapid increase in the human population. Based on the data he had at the time, Malthus predicted that the world population would increase exponentially, doubling every 40 years, thereby increasing at a faster and faster rate (40 years is the current doubling time of the world's population). If you start with the number 1 and keep doubling it, you will see an example of exponential growth.

The models of Fibonacci, Malthus and some other scientists all predict that the population will grow faster and faster. This is an alarming prospect, but does not seem to happen in experiments performed when there are limited resources, such as food and space to live in. Experiments with small animals and fungi in the laboratory, and with larger animals in fenced areas in the field show that as the resources start to run out, the reproduction rate reduces and the rate of growth slows down. The Belgian scientist Pierre Verhulst (1804 – 1849) while at the Belgian military school, the Ecole Royale Militaire, developed a model, called the logistic model, which took into account these observations. He introduced the idea of a 'carrying capacity' or maximum sustainable population that the environment will support.

We can illustrate the Malthus exponential model and the Verhulst logistic model by looking at the population of Australia since 1900. The small dark boxes on the graph below show the Australian Bureau of Statistics figures for the number of people in Australia (in millions) up until 2016. If we model these data with an exponential curve, we get the top curve in the figure. The middle curve is the logistic model. Both these curves fit the population numbers up to the present time well, but predict quite different future populations.



According to the exponential (Malthus) model, the population of Australia will continue to grow at a faster and faster rate, with a predicted population of about 99 million people in the year 2100, and about 504 million people in 2200. The logistic (Verhulst) model predicts that the population will keep on growing, but at a slower and slower rate; the predicted population in 2100 is about 49 million people; the population would level off eventually at about 63 million people.

The Australian Bureau of Statistics (ABS) uses a mathematical model to predict the population of Australia into the future to assist in planning for the number of people who will be living here. The predictions of their model are shown as the bottom curve in the figure. It has the shape of a logistic curve, but levels out much faster than the middle curve, predicting a population in 2100 of about 30 million people, and a maximum population of about 31 million.

Prediction is one powerful aspect of a mathematical model. By putting in the numbers we know, such as for the Australian population, we can predict what a population will be in the future, according to our model. Of course, the accuracy of our predictions depends on how good our model is, that is how well it describes the phenomena that affect population growth.

Another important use of a population model is to predict what will happen to the population if something changes, for example if the birth rate drops, if the number of immigrants is decreased, or if, say in an epidemic, many people die. Predicting changes in a population is particularly relevant to populations of animals, insects and plants which have become serious pests after being brought into Australia from overseas. These include rabbits (see the picture below), foxes, mice, cane toads and European carp among the animals, and prickly pear, Paterson's curse, salvinia, mimosa and scotch thistle, to name but a few of the plants. The populations of some of these have reached very high levels at times, causing serious problems for farmers and the environment.

Rabbits drinking at a waterhole before the introduction of the myxomatosis virus

How do we control such pests? Often there are a number of possible ways, but which one is best? Population models can be modified to include the effect of the release of a predator, the spread of a disease in the pest population, the effect of poisoning or some other control measure. It is then possible to use the mathematical models to predict what would happen to the population if the different control strategies were tried. The models can also be used to find the best way of carrying out a particular control measure. Sometimes the modelling is done together with small-scale experiments, but often only the model can be used because the experiments are too risky or too expensive.

In using a population model, we put the starting conditions and parameters (number of animals, how quickly they breed, etc) into our equations and predict the population at some later time. What if we change the starting conditions only slightly? We will end up with nearly the same final answer, right? Not necessarily. In some models, for example a variation on the Verhulst logistic model, with particular parameters, we find that the population does not change steadily towards some ultimate population, as we saw in modelling the Australian population, but changes rapidly and unpredictably with time. We say the model exhibits chaos: it loses its ability to predict, because a small change in the starting conditions produces a large change in how the population varies with time.

6.2 Population problems

Mathematically, the problems here are about *iteration* and about *exponential processes*.

Iteration is the process of carrying out the same operation over and over again. Let's take a simple example, that of multiplying by 2. Start with the number 1. Multiply it by 2 to give 2. Multiply the answer 2 by 2 again to give 4. Multiply 4 by 2 to give 8, and so on.

To do the calculation here: [1] [enter] [×] [2] [enter] [enter] ... to multiply by 2 each time. You'll have to keep count of how many times you have multiplied by 2.

Iterating by multiplying by a constant (2 here) is an example of an *exponential process*. You may have heard the term exponential growth, which many people interpret to mean 'grow quickly'. But exponential growth has a precise mathematical meaning and some interesting properties which we shall explore shortly.

Exponential iteration models a number of processes such as radioactive decay, population growth and absorption of light. If the constant we multiply by is larger than 1, we get exponential growth; if it is less than 1 (but greater than 0), we get exponential decay.

The use of scientific notation makes writing down our calculations much easier. For example, if we start with 5 and multiply it by 2 three times, we get $5\times2\times2\times2$, written as $5\times2^3=40$. 2^3 means three 2s multiplied together. If we multiply 5 by 2 ten times, we have $5\times2^{10}=5120$. 2^{10} means ten 2s multiplied together.

The exponentiation key is [∧] so, to calculate 2^3, we would press [2] [∧] [3] [enter].

The [EE] on the comma key gives 10^ so, to enter 2×10^6, press [2] [EE] [6].

Solutions to all problems are in Section 6.4.

6.2.1 Exponential iteration

Write down the results of the first 10 iterations of multiplying by 2, starting with 1.

6.2.2 Lots and lots of bacteria

Bacteria multiply (increase in number) by dividing — into two. One type of bacterium, *Streptococcus exponentiae*, divides every minute. If we start with 1 bacterium, it divides into 2 bacteria after 1 minute. Each of these 2 bacteria divides after 1 more minute, and so on. *The total number of bacteria grows exponentially.*

Make up a table with time in the first column and the number of bacteria in the second column. *How many bacteria are there after 10 minutes? after 20 minutes? after 1 hour? after n minutes? Why isn't the Earth covered metres deep in these bacteria?*

6.2.3 Malthus and exponential growth

Thomas Robert Malthus (1766 – 1834) made some worrying predictions for the world population, and his name is often associated with the idea of exponentially growing populations. Look up Malthus to find out the details of his ideas. Why was he worried about the world's population?

Malthus looked at the United States population to try to verify his ideas. He concluded the growth was exponential. *From the numbers in the table below, can you tell if he was correct for the years until he died?*[24] *What about the population growth after about 1860?*

Year	**Population** (millions)	**Year**	**Population** (millions)	**Year**	**Population** (millions)
1790	3.90	1860	31.4	1930	123
1800	5.30	1870	38.6	1940	132
1810	7.20	1880	50.2	1950	151
1820	9.60	1890	62.9	1960	179
1830	12.9	1900	76.0	1970	203
1840	17.1	1910	92.0	1980	227
1850	23.2	1920	106	1990	249

Why might the populations not continue to increase exponentially?

Note: The graphics-calculator programs in Section 6.5 make teachers' lives easier for this problem, and for modelling the Australian and world populations.

If students are to do the data plotting and curve fitting, they should do it manually rather than using a program.[25] The data (contained in the programs) could be transferred from the teacher's calculator.

PTO

[24] *Hint*: If the growth is exponential, each population should be a constant multiple of the previous value. Try a multiplier of 1.35, meaning the population increased by 35% every 10 years. The numbers you obtain only need to be close to the actual numbers, not exactly the same.

[25] See *Fitting Functions to Data 1*, Chapter 5

6.2.4 Cane toads

The Hawaiian cane toad *Bufo marinus* was introduced into Australia to control sugar-cane beetles. From the original 101 toads released in north Queensland in June 1935, the population grew rapidly and spread across the countryside. The table below shows the total land area of Australia colonised by cane toads for the years 1939 to 1974.

Year	**Area** ($10^3\,\text{km}^2$)	**Year**	**Area** ($10^3\,\text{km}^2$)
1939	33.8	1959	202
1944	55.8	1964	257
1949	73.6	1969	301
1954	138	1974	584

Is exponential growth a good model here? You can get a rough idea by the process we used for the Malthus data — finding ratios of successive values — but a plot of the data together with an exponential fit (graphics calculator required) will provide a better answer. *What is the exponential equation of best fit?*

Given that the area of Queensland is 1728 thousand km^2 and the area of Australia is 7619 thousand km^2 when, according to the exponential model, will (did) the cane toads colonise all of Queensland? all of Australia?

The cane growers were warned by Walter Froggart, President of the New South Wales Naturalist Society, that the introduction of cane toads was not a good idea and that the toads would eat or poison the native ground fauna. He was immediately denounced as an ignorant meddlesome crank. He was also dead right.

6.3 Other exponential problems

6.3.1 Piles of paper

A ream of paper (500 sheets) is about 50 mm thick, so that one sheet is about 0.1 mm thick. Take a sheet of paper, cut it in half and put the two halves one on top of the other. Cut this pile of 2 pieces in half and make a pile of 4 pieces. Keep cutting the pile in half and stacking the pieces up.

Now suppose you could make 42 cuts altogether (you'd need big scissors!). *How high would your final pile be?* Try making up a table like the one below to keep track of your pile. *Where would your pile reach to?* Kilometres might be a good unit to use eventually. *Write down a formula that tells you the height after n cuts. What units will you use?* Be careful!

Cut	**Height of pile**	
number	**in sheets**	**in mm**
1	2	0.2
2	4	0.4
3	8	0.8
4	16	1.6
⋮	⋮	⋮

This is an example of where maths lets you find an answer to something you can't actually do in real life.

6.3.2 Shoeing a horse

A rich man sends his horse to the blacksmith to have 4 new horseshoes put on. Each shoe needs 5 nails. The blacksmith offers to charge either $100 per nail (they're gold!) or 1c for the first nail, 2c for the second nail, 4c for the third nail, and so on, the cost doubling each nail. *Which offer should the rich man take?*

Think first which offer *you* would take. Then do some calculations. Don't forget to add up the total cost at the end. A calculator might be useful. *Did you pick the better offer? Was there much difference?*

Perhaps you might like to write down a function for the cost of the second offer after n nails and graph it. Write down a new version of the problem if each shoe needed 6 nails. What could the first offer be in this case?

6.3.3 Interest rates

Once you have some money in the bank, you start to think about interest, and you might want to answer a question like the one below to work out how much money you will have some time in the future.

> If the annual interest rate on a bank account is 12% compounded monthly and you deposit $10, how much money will you have after 1 year? after 5 years? after 10 years?

What does this mean? In simpler terms, it means that every month the bank will pay you an amount of interest equal to 1% (an annual interest rate of 12% means a monthly interest rate of $12\%/12 = 1\%$) of the amount you have in the account at the end of the month. So, after the first month, the bank will pay into your account 1% of $10 or $0.01 \times \$10 = \$0.10 = 10c$ in interest, and you will then have in your account

$$\$10 + \$0.10 = 1.01 \times \$10 = \$10.10.$$

After the second month, the interest will be 1% of $10.10 or $0.01 \times \$10.10 = \$0.101 = 10.1c$, and you will then have in your account

$$\$10.10 + \$0.101 = 1.01 \times \$10.10 = 1.01 \times (1.01 \times \$10) = \$10.201.$$

Although the bank won't pay you the 0.1c, they leave it in for future calculations.

Can you see a pattern? At the start of each month, the new amount in your account will be the amount you had last month times 1.01.

Now can you answer the question above? Can you write down a formula using exponential notation for the amount in your account after n months? How long before you have $15? $30?

6.4 Solutions to problems

Exponential Iteration

$1 \times 2 = 2 = 2^1$
$1 \times 2 \times 2 = 4 = 2^2$
$1 \times 2 \times 2 \times 2 = 8 = 2^3$
$1 \times 2 \times 2 \times 2 \times 2 = 16 = 2^4$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 = 32 = 2^5$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64 = 2^6$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128 = 2^7$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256 = 2^8$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 512 = 2^9$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1024 = 2^{10}$

Lots and Lots of Bacteria

You should end up with a table containing the numbers above and lots more. From your table, you can read off the answers to the questions. If you are using a calculator, it will probably switch to scientific notation when the numbers become large enough; powers of 10 are just like powers of 2, but much easier to write down. 10^3 is a 1 followed by three zeros, 10^{10} a 1 followed by 10 zeros, and so on.

Time	Number of Bacteria
after 1 minute	2
after 10 minutes	1024 (= 2^{10})
after 20 minutes	$1,048,576$ (= 2^{20})
after 1 hour	about 1.15×10^{18} (= 2^{60})

The number of bacteria after n minutes is 2^n.

The Earth isn't covered with these bacteria because in reality the growth is not exponential but has a limit: environmental conditions ultimately limit growth and scientists continue to discover new antibiotics.

Malthus and Exponential Growth

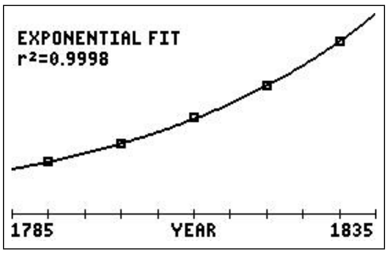
Thomas Malthus was worried about the world's population because he believed that population growth increased exponentially such as in the examples above, but that the food supply would only grow linearly (as a straight line — much more slowly). In other words, left unchecked, the human population would eventually exceed its food and other resources, leading to overcrowding, poverty, malnutrition, disease, crime and war. You can find out more about Malthus online.

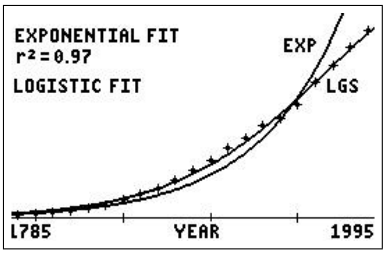
Starting at 1790, multiplying each number in the table by 1.35 gives a number close to the next number, up until 1860: the US population increased by about 35% every 10 years from 1790 to 1860. The 10-year growth rate then decreased to values in the range 20–30%, and finally down to around $9\frac{1}{2}$% in the decade to 1990. A marked decline in growth occurred between 1910 and 1950, during the two world wars and the Great Depression. The growth rate picked up a bit after the wars (the baby boom: 1950 – 60), then slowly declined again.

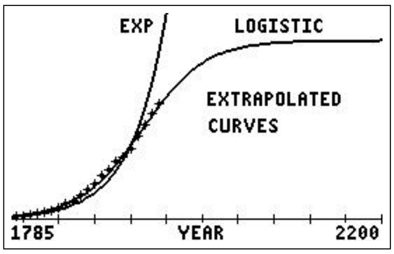
The left-hand figure below shows the exponential fit to the US population for the years 1790 – 1830. The fit is excellent (coefficient of determination r^2 very close to 1).

The middle figure shows exponential and logistic fits to the full data set 1785 – 1990. The exponential fit ($r^2 = 0.97$) is not quite as good now and doesn't follow the trend in later years. The logistic fit is much better.

The right-hand figure shows an extrapolation of the two curves to the year 2200. The exponential model predicts a US population in 2200 of more than 29 billion, the logistic model about 386 million, eventually stabilising at about 388 million.



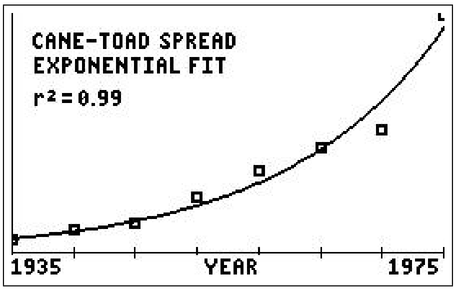




Cane Toads

The ratio of successive terms jumps about a bit, between 1.17 and 1.94, with a mean of 1.34. The population is therefore increasing roughly exponentially.

The exponential fit to the data looks reasonable (the value from 1969 is a bit low: a drought period from 1964 to 1969?). The curve of best fit is $y = 36.9 \times 1.08^t$ or, using *the* exponential function, $y = 36.9e^{0.0774t}$, where t is years since 1939.



Use your graphics calculator to find when $y = 1728$ and $y = 7619$ on the curve of best fit.[26] According to the exponential model, Queensland was overrun by cane toads between 1988 and 1989, Australia between 2007 and 2008. Clearly, and fortunately, there are some factors that restrict the spread of the cane toad. *Find out more?*

Piles of Paper

1 sheet of paper = 0.1 mm thick.

After 1 cut, 2 (2^1) sheets of paper = 0.2 mm thick.
After 2 cuts, 4 (2^2) sheets of paper = 0.4 mm thick.
After 3 cuts, 8 (2^3) sheets of paper = 0.8 mm thick.

⋮

After 42 cuts, 4,398,046,511,104 (2^{42}) sheets of paper $\approx$ 439,804,651,110 mm = 439,805 km thick — a little more than the distance from the Earth to the Moon.

[26] Graph $y = 1728$ and $y = 7619$, and use *intersect* in the calc menu to find in what year these lines intersect the curve of best fit.

Shoeing a Horse

A horse needs 4 new horseshoes with 5 nails in each.

Nail	**Cost of nail**
1st	1c
2nd	2c
3rd	4c
4th	8c
5th	16c
6th	32c
7th	64c
8th	\$1.28
9th	\$2.56
10th	\$5.12
⋮	⋮
19th	\$2,621.44
20th	\$5,242.88
Total	\$10,485.75

If the man pays \$100 per nail, it will cost him \$2,000 to shoe the horse.

If he pays by the nail as in the table above (the total cost is the sum of all the costs in the second column), it will cost him \$10,485.75, more than 5 times the flat rate. The choice is now obvious, and demonstrates clearly how rapidly exponential functions can increase.

Interest Rates

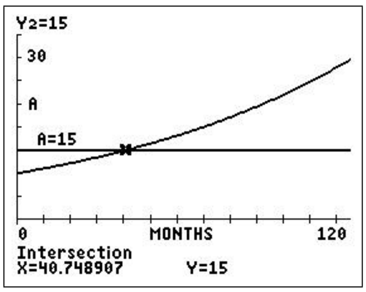
Deposit	**Annual Interest Rate**	**Time Period**	**Value**
\$10.00	12%	after 1 year	\$11.27 $(= \$10.00 \times 1.01^{12})$
	compounded monthly	after 5 years	\$18.17 $(= \$10.00 \times 1.01^{60})$
		after 10 years	\$33.00 $(= \$10.00 \times 1.01^{120})$

The formula for the amount A in a bank account, given compound interest, is

$$A(n) = D\left(1+\frac{i}{100}\right)^n,$$

where D is the initial deposit, i% the interest rate per month and n the number of months the money has been left in the bank.

It takes 41 months or 3 years 5 months before you have \$15, and 111 months or 9 years 3 months before you have \$30. Find these values by trial and error, by using the table feature on the graphics calculator or by graphing the function A with the number of months n as the independent variable, and finding where A equals 15 (figure below). See the next chapter for more on compound interest.



Amount in bank $A(n)$ versus month n, showing when the amount reaches \$15.

6.5 Population-modelling programs

These programs are available at *www.XXX*.

AUSPOP/AUSPOPCE: Australian population 1906 – 2016

This program (used for the first figure in the Introduction) plots the population of Australia, fits and plots an exponential function, a logistic function and a logistic curve from the Australian Bureau of Statistics (ABS). It also shows extrapolation of the curves to predict the Australian population in the future.

Use: Run the program. The screen tells you which keys can be used at any given time. After the program has finished, you can [trace] the graphs on the screen or manually replot the data (Plot1) and any of the models (Y_1: exponential; Y_2: logistic; Y_3: ABS) by turning on/off the appropriate plot/functions in [y=]. You can also change the [window]. Press [stat] Edit to see the data and model values.

MALTHUS/MLTHUSCE: US population 1790 – 2020

This programs plots the data for 1790 – 1830 from the table of US population on page 82 and fits an exponential function of the form $P(t) = ab^t$, where a and b are constants. You can do this manually on the calculator — the programs just makes it easier. You might like to work out possible values of a and b manually using just the first two data points.

The program also plots all the data from the table, and fits both an exponential function and a logistic function. It also shows extrapolation of the curves to predict the US population in the future.

Use: Run the program and choose which plot you want. The screen tells you which keys can be used at any given time. After the program has finished, you can [trace] the graphs on the screen or manually replot the data (Plot1) and any of the models (Y_1: exponential; Y_2: logistic) by turning on/off the appropriate plot/functions in [y=]. You can also change the [window]. Press [stat] Edit to see the data and model values.

WORLDPOP/WLDPOPCE: World population 1940 – 2020

This program (not used in the problems here) plots the world population, fits and plots an exponential function, a logistic function and a logistic curve from the US Bureau of Statistics (USBS). It also shows extrapolation of the curves to predict the world population in the future.

Use: Run the program. The screen tells you which keys can be used at any given time. After the program has finished, you can [trace] the graphs on the screen or manually replot the data (Plot1) and any of the models (Y_1: exponential; Y_2: logistic; Y_3: USBS) by turning on/off the appropriate plot/functions in [y=]. You can also change the [window]. Press [stat] Edit to see the data and model values.

After you have finished running any of these programs, delete lists FIT, EFIT, LFIT, POP, RESID *and* YR, *and clear functions* Y_1–Y_3. *Alternatively, run the program* POPCLEAR/ POPCLRCE.

7 Financial Mathematics 1: Compound Interest

7.1 Introduction

As with all mathematics, some simple interest calculations should be done by hand first, so that students understand what they are are calculating. Section 7.2, *Compound Interest: The Basics* (suitable for Year 9/10), leads on from hand calculations to interest calculations using the calculator. However, hand calculations very quickly become tedious once regular payments are involved; Section 7.3, *Using Sequences*, includes these in a logical development of the methods of Section 7.2.

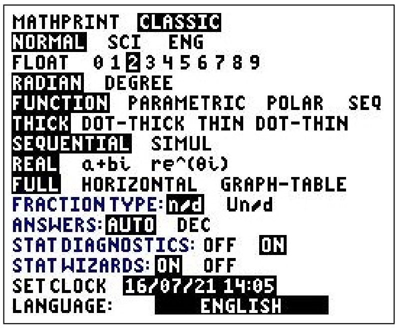
Financial Mathematics 2, in the third volume of this book, details the use of the calculator TVM solver for a variety of more-complex financial calculations.

7.2 Compound interest: The basics

Based on an article for students in Year 9 by Steve Arnold.

The TI-84 is a very powerful tool, more like a palm-top computer than a calculator. However, unlike Maths teachers, computers and calculators are very unforgiving. If you don't give them exactly the right information, you will probably get the wrong answer. So be careful!

The process — what you do — is crucial. Compare your results with the person beside you and ask the teacher if neither of you is sure.



Let's set the calculator to display numbers rounded to 2 decimal places, as we are working in dollars and cents. Press [mode]. Move the cursor to the third line, which sets the number of decimal places. Move the cursor to 2 and press [enter].

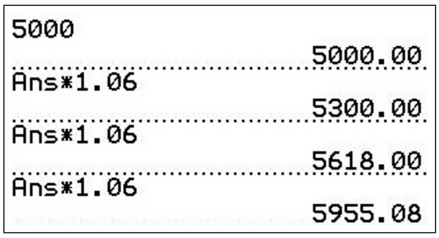
Press [quit] ([2nd] [mode]) to return to the Home screen.

Question 1 *Solutions to all questions in Section 7.4*

(a) If you invest \$5000 at an annual rate of 6% compounded annually, how much money will you have after 5 years? after 10 years?

Method A: Repeated multiplication

Type	See	Result
5000 [enter]	5000	5000.00
×1.06 [enter]	Ans∗1.06	5300.00
[enter]	Ans∗1.06	5618.00
[enter]	Ans∗1.06	5955.08
⋮	⋮	⋮



Don't forget to count the number of times you press [enter]: 1 press = 1 year.

(b) What calculation does the calculator perform each time you press [enter] (except the first time)?

(c) Write out the calculation steps as the calculator does them to find the amount of money after 5 years. Turn this into a formula involving a number raised to power 5 and hence do the calculation on the calculator the normal way to check your answer.

Method B: Using a formula

The compound-interest formula is

$$A = P\left(1 + \frac{R}{100}\right)^N,$$

where A is the amount of money after N years, P is the principal or starting amount and R is the annual interest rate expressed as a percentage.

For our question, $P = 5000$, $R = 6$ and $N = 10$.

On the Home screen, type the formula as 5000(1+6/100)^10 and press [enter].

```
5000(1+6/100)^10
                8954.238483
```

Question 2: How long does it take to double your money?

Clearly we could use Method A to answer this question by continuing to press [enter] until the result reaches 10,000. What about Method B? You have to find the smallest (integer) value of N that gives a value of A greater than 10,000.

(a) Press [entry] ([2nd] [enter]). This recalls the previous command. Change the value of N to one which you think will give an answer greater than 10,000 and press [enter] to re-calculate the formula.

Make up a table of the values of N you tried and the amount of money you found with each N. Identify which N answers the question and show that it does.

Method C: Using a table

Press the function-definition key [y=] and set Y1 = 5000(1+6/100)^X. Y1 represents A, the amount of money, and X (press [X,T,θ,n]) represents N, the time in years.

Now press [table] ([2nd] [graph]).

```
■\Y1■5000(1+6/100)^X
```

```
TABLE SETUP
 TblStart=0
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

X	Y1
0.00	5000.0
1.00	5300.0
2.00	5618.0
3.00	5955.1
4.00	6312.4
5.00	6691.1
6.00	7092.6
7.00	7518.2
8.00	7969.2
9.00	8447.4
10.00	8954.2

X=0

If your table does not start at X = 0 and go up in steps of 1, press [tblset] ([2nd] [window]) to go to the TABLE SETUP screen or table 'window'. With the cursor and [enter], set TblStart = 0 and ΔTbl = 1. Press [table] to return to the table.

(b) From the calculator table, by the end of which year does your money double? Scroll <u>down</u> in either column, <u>up</u> in the X column.

Question 3: If you invest \$5000 at 6% annual interest, compounded monthly, how long does it take to double your money?

```
■\Y1=5000(1+6/100)^X
■\Y2=5000(1+6/12/100)^(12X
)
```

An annual interest rate of 6% compounded monthly gives a monthly interest rate $R = 6/12/100$, with the time now in months.

X	Y1	Y2
0.00	5000.0	5000.0
1.00	5300.0	5308.4
2.00	5618.0	5635.8
3.00	5955.1	5983.4
4.00	6312.4	6352.4
5.00	6691.1	6744.3
6.00	7092.6	7160.2
7.00	7518.2	7601.8
8.00	7969.2	8070.7
9.00	8447.4	8568.5
10.00	8954.2	9097.0

X=0

The amount of money at the end of year X, month 12X, is then $A = 5000(1+6/12/100)^{12\text{X}}$.

Press [y=]; set Y2 = 5000(1+6/12/100)^(12X).

Now press [table] and look at the values in the Y2 column.

(a) In which year does the amount double now?

(b) Compare Y1 and Y2. What does each column represent? Which compounding method is better?

Note: From now on, we will just use Y2. Press [y=], move the cursor over the = sign of Y1 and press [enter] to turn it off. Press [table] to return to the table.

Question 4: If the annual interest rate is now 8% compounded monthly, in which year does the amount double?

X	Y2
0.00	5000.0
1.00	5415.0
2.00	5864.4
3.00	6351.2
4.00	6878.3
5.00	7449.2
6.00	8067.5
7.00	8737.1
8.00	9462.3
9.00	10248
10.00	11098

Y2=5000

Move the cursor to the table heading Y2; press [enter].

Change the 6 to 8 and press [enter] again. The table values will reflect the new interest rate.

Question 5: By the beginning of which month of the 9th year does the amount double if the annual interest rate is 8% compounded monthly?
Note that X = 0.00 is the beginning of the 1st year, so that X = 8.00 is the beginning of the 9th year.

In [tblset], set TblStart = 8 and ΔTbl = 1/12 to give monthly increments.

```
TABLE SETUP
 TblStart=8
 ΔTbl=1/12
Indpnt: Auto Ask
Depend: Auto Ask
```

X	Y2
8.00	9462.3
8.08	9525.4
8.17	9588.9
8.25	9652.8
8.33	9717.1
8.42	9781.9
8.50	9847.1
8.58	9912.8
8.67	9978.9
8.75	10045
8.83	10112

X=8

You will have to count down to find the month: 8.00 ≡ beginning of January; 8.08 ≡ beginning of February, etc.

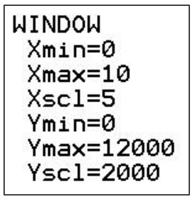
(a) At the beginning of which month does 8.33 correspond to?

(b) Find the answer to the question from the table.

Method D: Using a graph

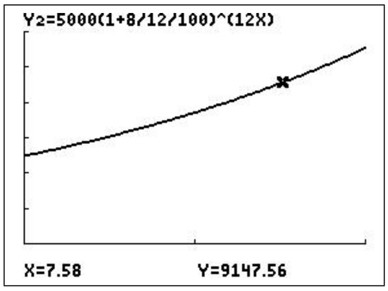
We already have the formula for the graph in Y2.

Press [window]. Here we have to tell the calculator how to set up the axes to view the graph. Put in the values shown.



Press [graph] to see the graph of Y2.

Press [trace] and use the left- and right-arrow keys to move along the curve.

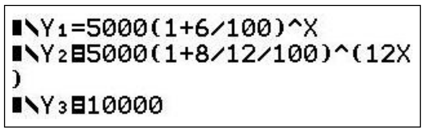


(c) In setting the [window], what does X represent? Why choose 0 for Xmin? What is the smallest number we could choose for Xmax? What does Y represent? What is the smallest number we could choose for Ymax?

(d) When you use [trace], unless you are lucky you won't find a point at which Y is exactly 10, 000. This is because the cursor jumps from pixel to pixel on the screen, rather than moving smoothly through all numbers. However, you can find points at which your money has at least doubled. Using the cursor, find the smallest value of X for which this is true. This is an approximation to the exact answer.

(e) If you move the cursor one pixel to the left (press the left-arrow key once) of the X value you found in (d), you can get some idea of the accuracy of your answer to the question. What are the X and Y values one pixel to the left of the X value you found in (e)? Between what times (in decimal years will do) does the exact answer then lie? You might like to think in terms like 'at this X, the Y value is just too large; at this X, the Y value is just too small'.

To find a more accurate answer, set Y3 = 10000, the amount of money we want to reach. Press [graph] or [trace] to display this second curve. We will calculate the approximate intersection point of the two curves, i.e. solve the equation Y2 = Y3, to find when the original amount doubles.

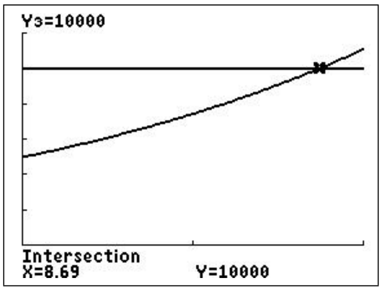


Press [calc] ([2nd] [trace]). Press [5] to select *intersect*.

The calculator now asks which curves you want to intersect (at other times, there may be more than two curves on the screen). The cursor should automatically be on Y2, the first function turned on in the function list. Press [enter] to select it. The cursor will now move to Y3. Press [enter] to select it.

The calculator now asks for a guess for the intersection point. Move the cursor somewhere near the intersection point and press enter.

Values are displayed to 2 decimal places because we set this in mode.



(f) The *intersect* operation just gives us a better approximation to the exact answer. From *intersect*, what is the answer to the question? Is it in dollars or years?

(g) How would you incorporate regular payments into Method A?

Use this to answer:

If you invest \$1000 at an monthly rate of 1%, compounded monthly, and deposit \$100 into the account every month (starting at the beginning of the second month), how much money will you have after 1 year?

7.3 Using sequences

This section illustrates the use of Sequence Mode on TI graphics calculators for simple financial calculations, including regular payments, following on from the basic calculations in Section 7.2. The use of (discrete) sequences for these sorts of calculations is more intuitive than the usual treatment using continuous functions.

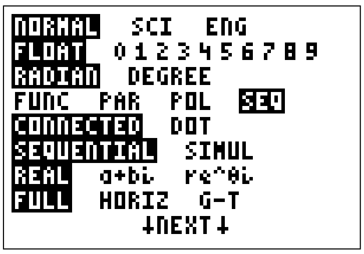
7.3.1 Sequence notation

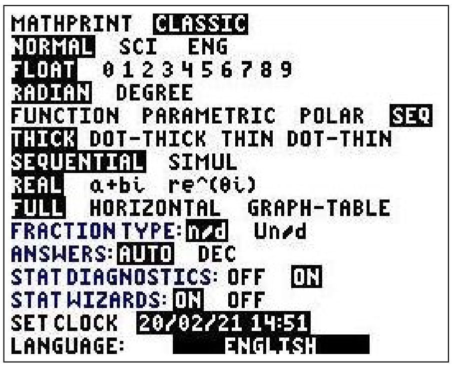
In the usual notation, a general sequence is written as $u_0, u_1, u_2, u_3, \ldots, u_n, \ldots$, where each term u_i is a number. The subscript gives the position of the term in the sequence.

The TI calculators use the notation $u(n)$ for the nth term of the sequence, rather than u_n, so the general sequence is written $u(0)$, $u(1)$, $u(2)$, $u(3)$, $\ldots$, $u(n)$, $\ldots$. This was presumably done for ease of display, but it reinforces an important fact about sequences: a sequence is really just a function with domain the positive integers, or some subset of them.

7.3.2 Using sequence mode

Sequences can be defined either recursively or explicitly, displayed and graphed in Sequence Mode. To select this, press mode and, with the cursor and enter, select *Seq* as shown below for the TI-84 Plus (left) and the TI-84 CE (right).





Seq mode is just one of four possible ways to define a function on TI calculators. Because a sequence is just a function as far as the calculator is concerned, all the calculator graphing keys (top row) are relevant.

Press the function-definition key [y=] to see the three available sequences, u, v and w.

On TI-84Plus calculators, the nth term, $u(n)$, can be written as any combination of n, $u(n–1)$, $u(n-2)$, $v(n-1)$, $v(n-2)$, $w(n-1)$ and $w(n-2)$. There is more flexibility on a TI-84CE.

Regular deposits

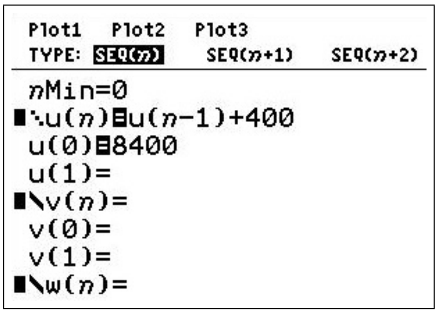
Example 1: Initial amount \$8400; deposit \$400 per time step (n increasing by 1: week, fortnight, month or year).

If u_n is the amount of money (\$) in the account at time (step) n: $u_0 = 8400$; $u_n = u_{n-1} + 400$. The amount at time step n is the amount at time step $n-1$ plus 400.

On the calculator: $u(0) = 8400 \quad u(n) = u(n-1) + 400$.

Set this sequence up on your calculator as shown below. The independent variable here, n, is produced by the [X,T,θ,n] key. The sequence name u is [2nd] [7].

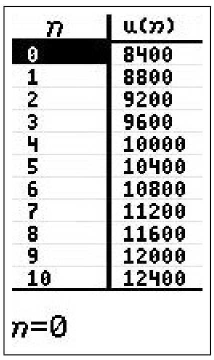
On a TI-84 Plus (below left), the initial value $u(n\text{Min})$, 8400, must be in curly brackets (on the bracket keys). These are not required for $u(0)$ on the TI-84 CE (below right).



While we are on this screen, check that you have three dots to the left of the definition of $u(n)$, indicating that only the sequence points will be plotted (no joining lines) when we graph the sequence. If not, move the cursor here and press [enter] several times until the three dots come up (on an 84Plus), or press [enter] and use the arrow keys to select this (CE).

Table: Now we use the table feature of the calculator to display the values of the sequence.

First a table 'window': press [tblset] ([2nd] [window]), set TblStart = 0 and ΔTbl = 1.



Press [table] ([2nd] [graph]). Scroll down in either column to see the terms of the sequence. If you want to scroll back up past the top of the screen (after you have scrolled down), you have to be in the n column (you will get ERROR here for $u(-1)$).

Home screen: Another way of displaying terms of a sequence u is to type on the Home screen a command of the form $u(\textit{start, end}\ [, \textit{increment}])$. For example, typing $u\,(1, 9, 2)$ and pressing [enter] will display $u(1)$, $u(3)$, $u(5)$, $u(7)$ and $u(9)$. *increment* is optional: a value of 1 assumed if it is not entered.

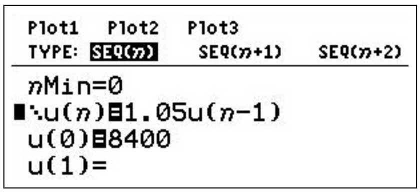
To display the value of just one term, type $u(n\,\textit{value})$, for example $u(6)$, and press [enter].

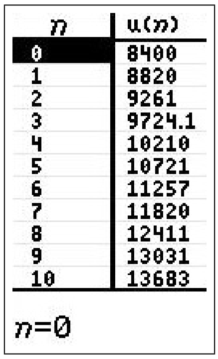
Compound interest

Example 2: Initial amount \$8400; compound interest 5% per time step.

If u_n is the amount of money (\$) in the account at time (step) n: $u_0 = 8400$; $u_n = 1.05u_{n-1}$. The amount at time step n is the amount at time step $n-1$ plus 5% of that amount.

On the calculator: $u(0) = 8400$ $\quad u(n) = 1.05u(n-1)$.



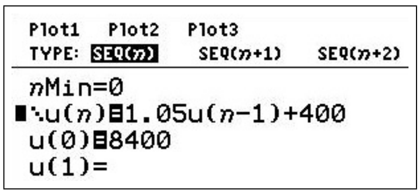


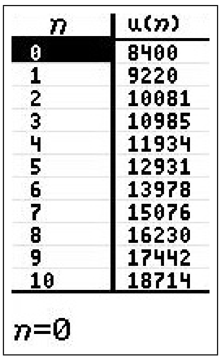
Compound interest and regular deposits

Example 3: Initial amount \$8400; compound interest 5% and deposit of \$400 per time step.

If u_n is the amount of money (\$) in the account at time n: $u_0 = 8400$; $u_n = 1.05u_{n-1} + 400$. The amount at time step n is the amount at time step $n-1$ plus 5% of that amount plus 400.

On the calculator: $u(0) = 8400$ $\quad u(n) = 1.05u(n-1) + 400$.



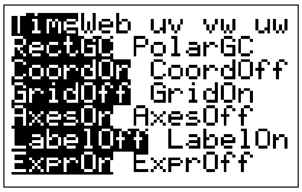


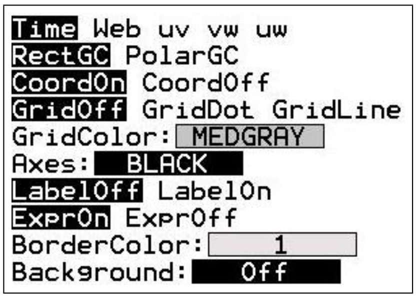
Exercise: (Question 5(g), page 99) If you invest \$1000 at an monthly rate of 1%, compounded monthly, and deposit \$100 into the account every month, how much money will you have after 1 year?

Solution on page 99.

Graphing the sequence

There are several ways to graph sequences, as shown when you press [format] ([2nd] [zoom]): there's an extra line here (the top one) in Seq mode.

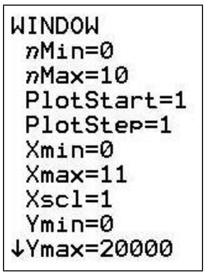


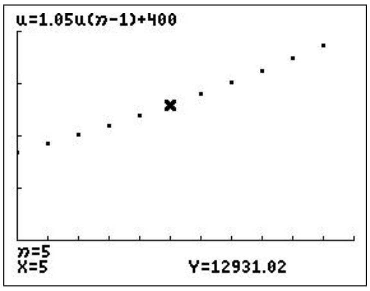


Time plots $u(n)$ against n, which is what we want here. Select it if necessary with the cursor and [enter].

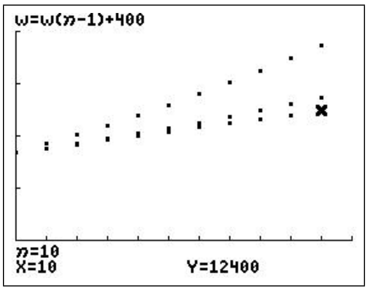
Next set a graph window: press [window] and you will see a few more parameters than usual. In our *Time* plot, n is plotted along the X axis, so the n range should be contained within the X range.

Set up your window as shown below (Yscl is 5000) and press [trace]. Use the arrow keys to move along the points of the sequence.





The plot below shows all three sequences: u_n, with both compound interest and regular deposits, at the top; v_n, with just interest in the middle; and w_n, with just regular deposits, at the bottom.



Shortcut: In all plots, specify nMax, then press [zoom] [0] (*ZoomFit*); this sets appropriate X and Y scales. You may want to change these a bit in [window] after the graph has been plotted. The other Zoom options, such as *ZoomIn* and *ZoomBox*, work too.

7.4 Solutions

Note for teachers

Before starting the calculations here, students should have done some basic compound-interest calculations by hand. We use the calculator to be able to answer questions about compound interest that would take a long time by hand.

Some of the questions are designed to make students think about their use of the calculator as a tool. This is important, but clearly the questions can be chosen/varied to match the ability of the class.

Question 1

(a) If you invest \$5000 at an annual rate of 6% compounded annually, how much money will you have after 5 years? after 10 years?

After 5 years, you will have \$6691.13, and after 10 years, you will have \$8954.24, both rounded to the nearest cent.

(b) What calculation does the calculator perform each time you press enter (except for the first time)?

The calculator multiplies the previous answer/result by 1.06.

(c) Write out the calculation steps as the calculator does them to find the amount of money after 5 years. Turn this into a formula involving a number raised to power 5 and hence do the calculation on the calculator the normal way to check your answer.

The calculation is

$$5000 \times 1.06 \times 1.06 \times 1.06 \times 1.06 \times 1.06 = 5000 \times 1.06^5 = 6691.13,$$

rounded to 2 decimal places.

The calculator should not be a 'black box': students need to understand what it is actually doing.

Question 2

How long does it take to double your money?

(a) Make up a table of the values of N you tried and the amount of money you found with each N. Identify which N answers the question and explain why it does.

Clearly values for N in a table will vary, but eventually they should find that $N = 12$ does the trick. $N = 12$ is the smallest (integer) value of N for which the amount of money is greater than or equal to $10,000$.

(b) From the calculator table, by the end of which year does your money double?

The amount doubles by the end of the 12th year.

Question 3

(a) In what year does the amount double now?

The amount still only doubles by the end of the 12th year.

(b) Compare Y1 and Y2. What does each column represent? Which compounding option is better?

Y1 is the amount of money when the interest is compounded annually, Y2 the amount of money when the interest is compounded monthly.

Compounding monthly gives a greater amount than compounding yearly at any given time, and so is the better option.

Question 4

If the annual interest rate is 8% compounded monthly, in which year does the amount double?

The amount now doubles by the end of the 9th year.

Question 5

By the beginning of which month of the 9th year does the amount double if the annual interest rate is 8% compounded monthly?

(a) At the beginning of which month does 8.33 correspond to?

The beginning of May.

(b) Find the answer to the question from the table.

The amount has doubled when $X = 8.75$, corresponding to the beginning of October of the 9th year. The fact that the 9th year starts when $X = 8.00$ may cause some confusion here.

(c) In setting the [window], what does X represent? Why choose 0 for Xmin? What is the smallest number we could choose for Xmax? What does Y represent? What is the smallest number we could choose for Ymax?

X corresponds to time in years. Xmin is the starting time value, hence 0. Xmax must be some number larger than 9, because we know from previous work, doubling occurs in the 9th year.

Y corresponds to the amount of money in dollars. Ymax has to be some number greater than $10,000$, because this is the amount we are aiming at. With some experimentation, we find that $12,000$ leaves room at the top for the function formula (not needed on a CE).

(d) When you use [trace], unless you are lucky you won't find a point at which Y is exactly $10,000$. This is because the cursor jumps from pixel to pixel on the screen, rather than moving smoothly through all numbers. However, you can find points at which your money has at least doubled. Using the cursor, find the smallest value of X for which this is true. This is an approximation to the exact answer.

Using [trace], the smallest value of X for which $Y \geqslant 10,000$ is $X = 8.72$ (84 and CE).

(e) If you move the cursor one pixel to the left (press the left-arrow key once) of the X value you found in (e), you can get some idea of the accuracy of your answer to the question. What are the X and Y values one pixel to the left of the X value you found in (e)? Between what times (in decimal years will do) does the exact answer then lie? You might like to think in terms like 'at this X, the Y value is just too large; at this X, the Y value is just too small'.

TI-84: For the pixel one to the left of the X value in (d), $X = 8.62$ and $Y = 9939.45$. The exact X value (time in years) therefore must lie between 8.62 ($Y < 10,000$) and 8.72 ($Y > 10,000$).

TI-84CE: One trace pixel to the left of the X value in (d) gives $X = 8.64$ and $Y = 9954.79$. The exact X value (time in years) therefore must lie between 8.64 ($Y < 10,000$) and 8.72 ($Y > 10,000$).

(f) The *intersect* operation just gives us a better approximation to the exact answer. From *intersect*, what is the answer to the question? Is it in dollars or years?

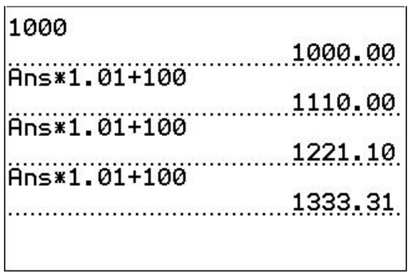
According to *intersect*, $Y = 10,000$ when $X = 8.69$ (rounded to 2 decimal places). This is in September, so the answer still remains 'by the beginning of October'.

(g) How would you incorporate regular payments into Method A?

Multiply by 1.01 and add 100.

If you invest $1000 at an monthly rate of 1%, compounded monthly, and deposit $100 into the account every month (starting at the beginning of the second month), how much money will you have after 1 year?

Month	**Type**	**See**	**Result**
0	1000 [enter]	1000	1000.00
1	×1.01+100 [enter]	Ans∗1.01+100	1110.00
2	[enter]	↓	1221.10
3	[enter]		1333.31
4	[enter]		1446.64
5	[enter]		1561.11
6	[enter]		1676.72
7	[enter]		1793.49
8	[enter]		1911.42
9	[enter]		2030.54
10	[enter]		2150.84
11	[enter]		2272.35
12	[enter]		2395.08



After 1 year, you would have $2395.08.

8 Probability and Statistics 1: Descriptive Statistics

8.1 Introduction

8.1.1 Probability and Statistics

The most comprehensive book I have seen on this topic is *Probability and Statistics with the TI-83 Plus: For A-Level Mathematics* by Peter Jones and Chris Barling.[27] Sadly, this is no longer readily available; if you have a copy, treasure it. They explain everything in great detail, with large numbers of screen shots. The exposition here is somewhat briefer but I'll follow their general outline, and even borrow some of their examples (labelled JB). Most of the exercises are also from there.

Some of the examples and exercises are from the course notes for a first-year introductory Statistics course by Dr Leesa Sidhu of UNSW Canberra.

What follows is not a textbook on Probability and Statistics (you'll need one) but how to do the various operations in such a course on a TI-84/CE graphics calculator. This replaces the various tables usually used in such courses, and makes many of the calculations quite simple. As always, students should do all these operations by hand first, so that they understand the process but, ultimately, always having to do the calculations manually stands in the way of doing any meaningful modelling and interesting applied problems.

Probability and Statistics 2, in the third volume of this book, deals with discrete and continuous probability distributions, and hypothesis testing.

8.1.2 Calculator notes

There are four options on the calculator that can help with the calculations here.

MATHPRINT

In CLASSIC mode, commands are typed on one line, with arguments in brackets. In MATHPRINT mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for the inputs. This really only applies to the powers and roots (blue) commands on the keyboard and to many of the commands in the [math] NUM menu, and so is of limited use in the probability and statistics operations here; where it applies, it is illustrated with a screen shot.

Set CLASSIC or MATHPRINT in [mode].

Catalog Help

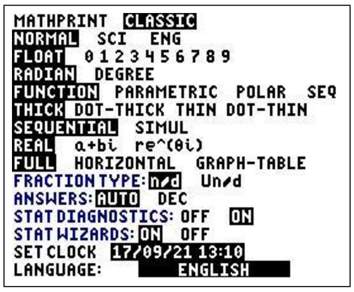
An app on the TI-84Plus,[28] built in to the operating system of a TI-84CE, this tells you what arguments are required for most of the commands on the calculator in CLASSIC mode, allows you to input the arguments and paste the whole command onto the Home screen. Activated by scrolling down to the particular command and pressing [+].

[27] Cambridge University Press, 2002, ISBN 0 521 52531 4; I have a copy.

[28] download from *education.ti.com/en/software/search*

STAT WIZARDS

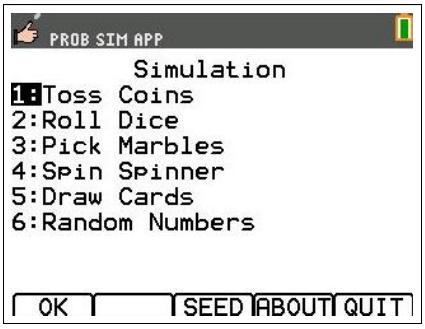
An alternative to *Catalog Help*, and much more useful here than MATHPRINT, this provides a menu for inputs to most of the commands and tests in stat CALC and stat TESTS, rather than having to type these into a one-line command in the correct order. It also works for some operations in the math PROB menu. Turn STAT WIZARDS on in mode and leave on. It is used in most of the examples here.



Make sure STAT DIAGNOSTICS is turned on too.

Prob Sim app

This app simulates a number of different ways of obtaining random data normally done with physical materials (see the figure). You can simulate single or multiple occurrences, with the outcomes summarised in on-screen histograms. *Random Numbers* can be used to simulate Lotto draws. Lots of data can be generated and stored in lists on the calculator for subsequent analysis. You can even change the probability of an event occurring. Highly recommended.



The app can be downloaded from *education.ti.com.* Click on *Downloads* and follow through to the app. You'll need TI Connect CE (for calculators with a mini USB port) or TI Connect to copy the app to your calculator.

Three associated activities are in the *Volume 1 Supplement: Activities for Years 9 and 10.*

- *Reaction Times and Statistics*: Programs are used to measure reaction times in various scenarios including simulated braking in a car. The data are displayed as box-and-whisker plots for subsequent analysis.
- *Statistics from Birthdays*: Class data on day and month of birth are used to provide an introduction to data presentation on a graphics calculator.
- *Probably Finding π*: An experimental-probability method for finding π.

8.2 Descriptive statistics

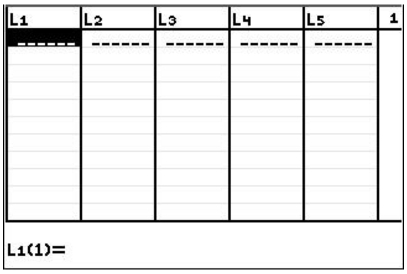
8.2.1 Data management

Example: To do any statistical analysis on the calculator, you need some data. These are stored in lists, either entered manually using the stat list editor or downloaded in a suitable format.[29] Here's an example of some data, pollution levels measured in six cities (JB).

City number	1	2	3	4	5	6
Pollution level (mg/m^3)	12	43	27	38	21	24

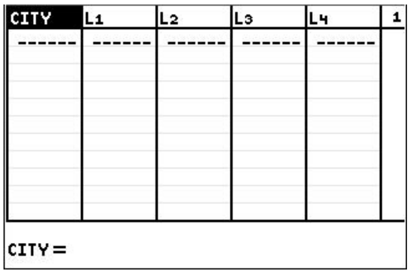
The default lists on the calculator are L_1 – L_6 (2nd (blue) characters on the corresponding number keys). Here we follow Jones and Barling by giving our lists names (up to 5 characters) that give some idea of what they are about.

Entering data into a named list



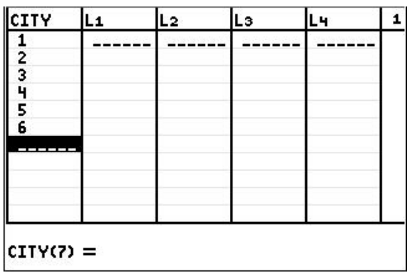
Create a list called CITY as follows.

Press [stat] and choose EDIT Edit... by pressing [enter] or [1].

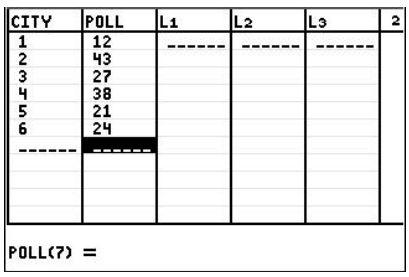


Move the cursor to the heading L_1, press [ins] and, using [A-lock] ([2nd] [alpha])[a], type in the word CITY. Press [enter] to complete the naming.

[a]on an 84, the cursor is already in A-LOCK



Enter the data into CITY: press the down arrow to move the cursor into the list; type in the data, pressing [enter] after each value.

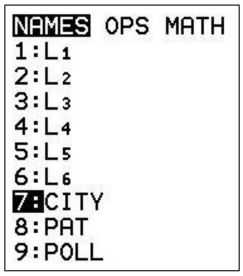


Exercise: Follow the steps above to enter the pollution data into a list called POLL.

Make sure you enter these values in the same order as those in CITY.

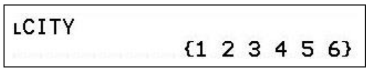
[29]using TI-Connect or TI-Connect CE

Displaying a list on the Home screen



Move to the list (2nd stat) NAMES menu. Type in the number against the list name you want to display or scroll down and press enter. This puts the list name onto the Home screen, preceeded by a small L to indicate it is a list.

Press enter to display the contents of the list.



To delete a list from the stat list editor menu, move the cursor to the list heading in stat EDIT Edit... and press del. The list will still be in the NAMES menu.

To delete a list from the memory, press mem (2nd +) 2 (Mem Management/Delete) 4 (List...), scroll down to the list name and press del.

8.2.2 Univariate data

Displaying the data

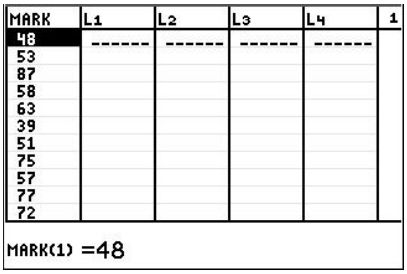
1. Histograms

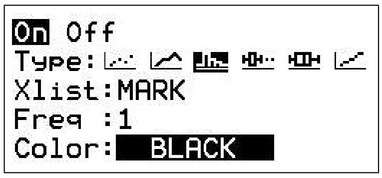
Example: The following data are marks in a Maths test out of 100.

48 53 87 58 63 39 51 75 57 77 72 62 47 68 72 38 73 49 74 80 50 86 73 77

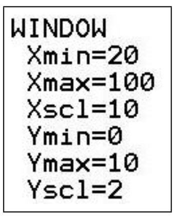
First enter the data into a list MARK, as we did in Section 8.2.1 (below left).

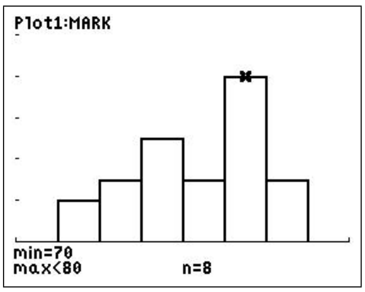
Press stat plot (2nd y=), select Plot1 with enter, and use the cursor and enter to set up the plot, as shown in the figure below right. MARK comes from the list NAMES menu.

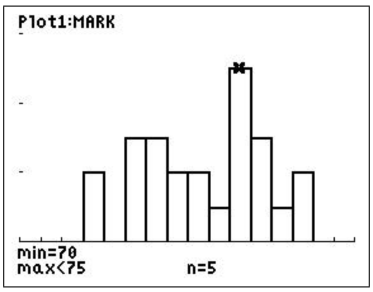




As we are going to plot a graph, we need to set a window. **The bin width is set by Xscl.**







In the histogram plots, trace has been pressed to draw the histogram, and the cursor moved to the interval with the highest frequency. For the right-hand figure above, the bin width Xscl has been changed to 5 and Ymax adjusted accordingly.

Exercises *Solutions in Section 8.3*

1. The life expectancies (in years) of people in 25 countries are listed below.

   58 65 68 74 73 73 75 71 72 61 67 66 37 50 66 64 72 74 48 41 44 44 49 48 48

   Construct histograms on your calculator with the first class interval starting at 35 and the last class interval ending at 80 using interval widths of: **(a)** 2; **(b)** 5; **(c)** 15.

2. The data below give the wrist circumference (in cm) of 15 men.

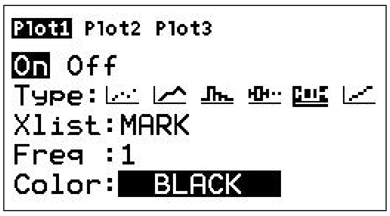
   16.9 17.3 19.3 18.5 18.2 18.4 19.9 16.7 17.1 17.6 17.7 16.5 17.0 17.2 17.6

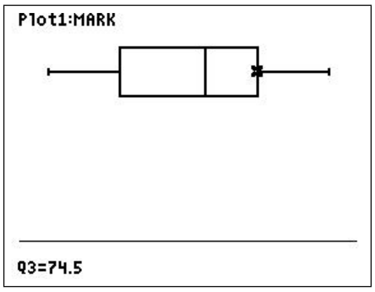
   Construct histograms on your calculator with the first class interval starting at 16.5 and the last class interval ending at 20.5 using interval widths of: **(a)** 0.5; **(b)** 1.0; **(c)** 2.0.

2. Boxplots

Set up a boxplot for the MARK data above using [stat plot] Plot1, as shown in the left-hand figure below. The easiest way to obtain a sensible window for statistics plots[30] is [zoom] [9] (ZoomStat).

In the right-hand figure below, [trace] has also been pressed, and the cursor moved to Q3.





Exercises

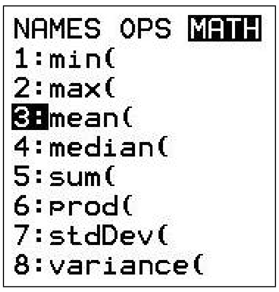
Construct boxplots for the two histogram exercises on the previous page.
Solution on page 110.

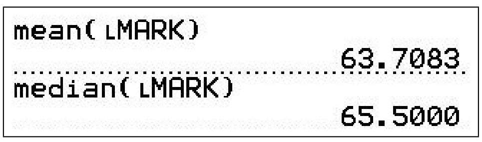
Summarising the data

1. Measures of centre

The calculator can calculate the **median** and **mean** of data in a list. The relevant operations are in the [list] MATH menu. *Values here are rounded to 4 decimal places.*

For the MARK data, use the relevant operations to calculate the median and mean.





[30] with the possible exception of histogram plots, because the bin width Xscl usually has to be set by hand

2. Variability

The TI-84 can calculate **variance** and **standard deviation**.

Here, we use the MARK data again. Values here are again rounded to 4 decimal places.

Finding the variance and standard deviation of the data in a list is just a matter of choosing the relevant command from the list MATH menu and the list name from the list NAMES menu.

```
NAMES OPS MATH
1:min(
2:max(
3:mean(
4:median(
5:sum(
6:prod(
7:stdDev(
8:variance(
```

```
variance(ʟMARK)
                    207.7808
stdDev(ʟMARK)
                     14.4146
```

3. 1-Var Stats command

You can also calculate the various statistics for a set of data in a list using the 1-Var Stats command in the stat CALC menu (below top left).

The *Catalog Help* screen for the *1-Var Stats* command, with the list name inserted from the list NAMES menu, is shown below top centre. This screen comes from pressing + with the cursor on the command (below top left).

The command from the *Catalog Help* screen pasted on to the Home screen (or the result of selecting the command directly from the stat CALC menu and inserting the list name), 1-Var Stats ʟMARK is shown below top right. Press enter to execute the command.

The input menu for the *1-Var Stats* command when STAT WIZARDS is turned on is shown below bottom left with the list name inserted from the list NAMES menu,. This screen comes from pressing enter with the cursor on the command (below top left). Move the cursor to *Calculate* and press enter to execute the command.

```
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7:QuartReg
8:LinReg(a+bx)
9↓LnReg
```

```
CATALOG HELP          ↑
1-Var Stats ʟMARK

[Xlistname,freqlist]

                      ↓
               PASTE ESC
```

```
1-Var Stats ʟMARK
```

```
   1-Var Stats
List:MARK
FreqList:
Calculate
```

```
   1-Var Stats
 x̄=63.70833333
 Σx=1529
 Σx²=102189
 Sx=14.41460361
 σx=14.11110428
 n=24
 minX=38
↓Q1=50.5
```

```
   1-Var Stats
↑Sx=14.41460361
 σx=14.11110428
 n=24
 minX=38
 Q1=50.5
 Med=65.5
 Q3=74.5
 maxX=87
```

The bottom two right-hand screens above show the results of the *1-Var Stats* command.

Exercise *Solutions in Section 8.3*

The weekly amounts (in \$) spent on food by 10 households were as follows:
170 123 87 98 112 150 98 134 106 114.
Find:

(a) the mean amount spent on food each week (to the nearest dollar);

(b) the standard deviation of the mean σ_x;

(c) the unbiased estimate of the population variance s_x^2;

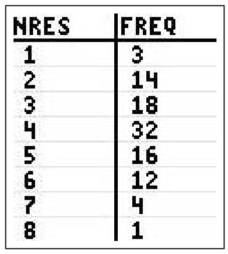
(d) the median amount spent on food each week;

(e) the interquartile range $Q_3 - Q_1$.

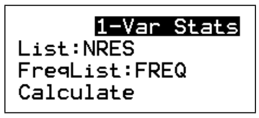
4. Grouped data

Example: One hundred households were surveyed, and the number of persons normally in residence recorded in tabular form as shown in the table (JB).

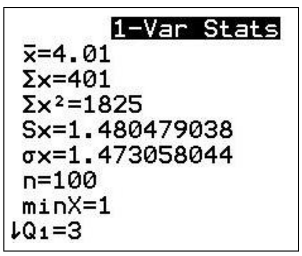
Number of residents	Frequency
1	3
2	14
3	18
4	32
5	16
6	12
7	4
8	1

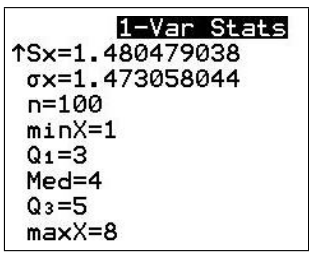


The statistics here are calculated using the *1-Var Stats* command. Enter the number of residents in list NRES and the frequency into list FREQ (Section 8.2.1).
Execute the *1-Var Stats* command with the two lists specified as shown (see the previous section on the different ways to do this).



The results are shown below.





Exercises *Solutions in Section 8.3*

1. One hundred households were surveyed, and the number of children normally in residence recorded in tabular form as shown in the table (JB).

Number of children	**Frequency**
0	14
1	23
2	31
3	18
4	12
5	0
6	2

Find:

(a) the mean number of children per residence;

(b) the standard deviation of the mean σ;

(c) the unbiased estimate of the population variance s_x^2;

(d) the median number of children per residence;

(e) the interquartile range $Q_3 - Q_1$.

2. The life (in hours) of 11 batteries is: 30 31 38 35 36 60 40 31 33 62 43. Generate a boxplot with outliers (the boxplot in [stat plot] with a dot on the right) of the data and write down all the associated values (using [trace]).

PTO

8.2.3 Bivariate data

Displaying the data

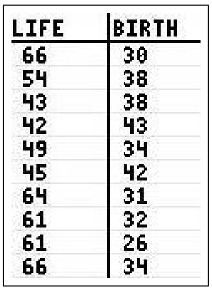
Scatterplots

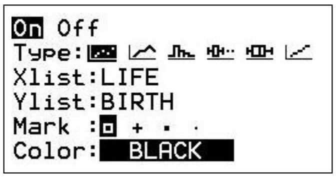
Example: The table below shows the life expectancy (in years) and birthrate (per thousand) of people living in ten countries (JB).

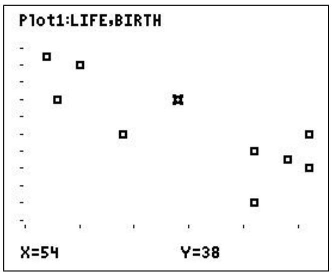
Life expectancy (years)	**Birthrate** (per 1000)
66	30
54	38
43	38
42	43
49	34
45	42
64	31
61	32
61	26
66	34

Enter the data into lists LIFE and BIRTH, respectively (Section 8.2.1).

Press [stat plot] ([2nd] [y=]), select Plot1 with [enter], and use the cursor and [enter] to set up the plot, as shown in the figure below centre. Press [zoom] [9] (ZoomStat) to set a suitable window and plot the graph. [trace] has been pressed after graphing to show the value at X = 54.



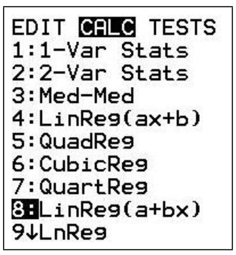


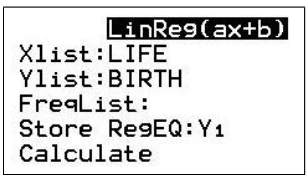


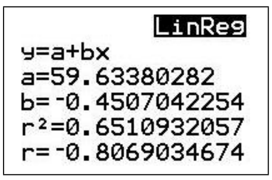
Summarising the data

1. Determining the equation of the least-squares line

In the [stat] CALC menu, select LinReg (a+bx) to calculate the least-squares fit of a straight line to the data. Add the two list names from the [list] NAMES menu. If you then add Y_1 from the [vars] Y-VARS Function menu, the equation of the line will be pasted in there for later graphing.



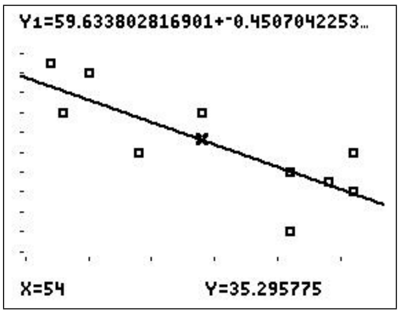




Here, the line of best fit is BIRTH $= 59.6 - 0.451$ LIFE (to 3 significant digits), with Pearson coefficient $r = -0.81$ and the coefficient of determination $r^2 = 0.65$.

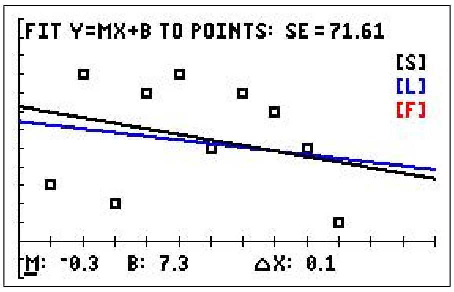
2. Plotting the least-squares line on a scatterplot

Just press graph or trace to plot the data and the least-squares line of best fit. In trace, you can read off values.



SPLSQFIT/SPLSFTCE program

The program generates points (or you can input your own) in lists L_1 (X) and L_2 (Y). You have to fit a straight line Y = MX + B to the points by varying M and B on-screen to minimise the squared error SE, which is displayed on the screen. The program will also plot the linear-regression line of best fit using the calculator command *LinReg* $(ax+b)$ and the median-median line of best fit using the calculator command *Med-Med*.



Main screen of the program showing the line fitted by adjusting M and B at the bottom of the screen (black line) and the linear-regression line of best fit from the calculator (blue line).

Exercise *Solutions in Section 8.3*

The table shows the life expectancies of males and females in the period 1900 to 1990.

Year	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990
Males	52	54	56	60	65	66	67	67	70	72
Females	55	57	63	67	69	73	74	74	76	78

(a) Construct a scatterplot with female life expectancy on the vertical axis and male life expectancy on the horizontal axis. Determine

(i) the equation of the least-squares line for these data;

(ii) the Pearson correlation coefficient r and the coefficient of determination r^2.

(b) Plot the least-squares line on the scatterplot.

(c) Predict female life expectancy when male life expectancy reaches 80.

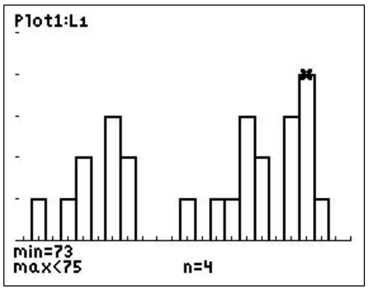
8.3 Solutions to exercises

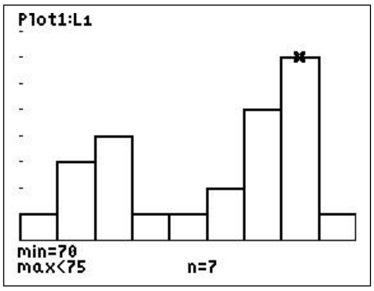
Exercises page 104

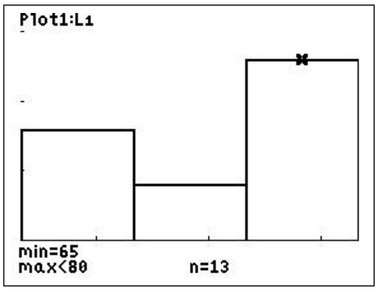
1. The life expectancies (in years) of people in 25 countries are listed below.

58 65 68 74 73 73 75 71 72 61 67 66 37 50 66 64 72 74 48 41 44 44 49 48 48

Construct histograms on your calculator with the first class interval starting at 35, the last class interval ending at 80 and using interval widths of: **(a)** 2; **(b)** 5; **(c)** 15.



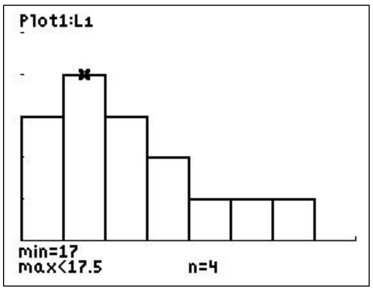


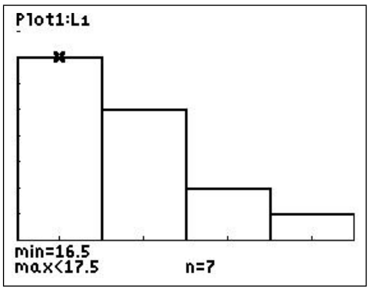


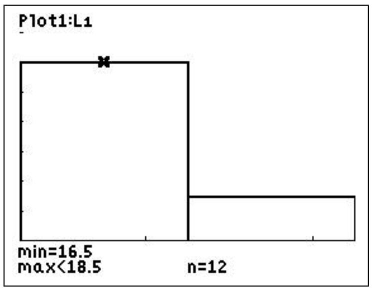
2. The data below give the wrist circumference (in cm) of 15 men.

16.9 17.3 19.3 18.5 18.2 18.4 19.9 16.7 17.1 17.6 17.7 16.5 17.0 17.2 17.6

Construct histograms on your calculator with the first class interval starting at 16.5 and the last class interval ending at 20.5 using interval widths of: **(a)** 0.5; **(b)** 1.0; **(c)** 2.0.



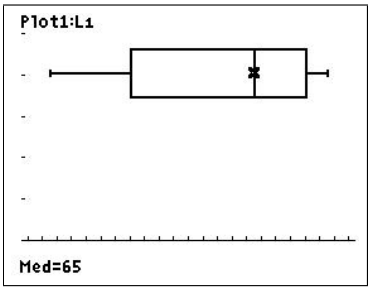




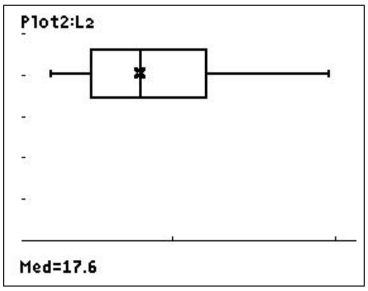
Exercises page 104

Construct boxplots for the two histogram exercises above.

1.



2.



Exercise page 106

The weekly amounts (in \$) spent on food by 10 households were as follows:
170 123 87 98 112 150 98 134 106 114.

Using 1-Var Stats:

(a) the mean amount spent on food each week (to the nearest dollar) is \$119;

(b) the standard deviation of the mean σ is \$24;

(c) the unbiased estimate of the population variance s_x^2 is $25.716^2 = 661$;

(d) the median amount spent on food each week is \$113;

(e) the interquartile range $Q_3 - Q_1$ is \$36 ($Q_3 = 134$).

```
1-Var Stats
x̄=119.2
Σx=1192
Σx²=148038
Sx=25.71553789
σx=24.3959013
n=10
minX=87
↓Q1=98
```

Exercises page 107

1. One hundred households were surveyed, and the number of children normally in residence recorded in tabular form as shown in the table (JB).

Number of children	Frequency
0	14
1	23
2	31
3	18
4	12
5	0
6	2

Using 1-Var Stats:

(a) the mean number of children per residence is 2.0;

(b) the standard deviation of the mean σ is 1.3;

(c) the unbiased estimate of the population variance s_x^2 is $1.34^2 = 1.8$;

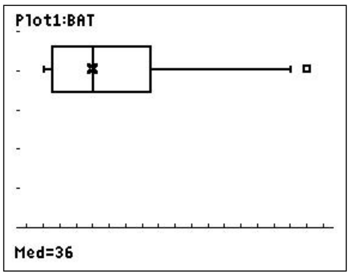
(d) the median number of children per residence is 2;

(e) the interquartile range $Q_3 - Q_1$ is 2 ($Q_3 = 3$).

```
1-Var Stats
x̄=1.99
Σx=199
Σx²=573
Sx=1.337078075
σx=1.330375887
n=100
minX=0
↓Q1=1
```

2. The life (in hours) of 11 batteries is: 30 31 38 35 36 60 40 31 33 62 43.

Generate a boxplot with outliers (the boxplot in [stat plot] with a dot on the right) of the data and write down all the associated values (using [trace]).




```
1-Var Stats
x̄=39.90909091
Σx=439
Σx²=18769
Sx=11.17546013
σx=10.65538315
n=11
minX=30
↓Q1=31
```

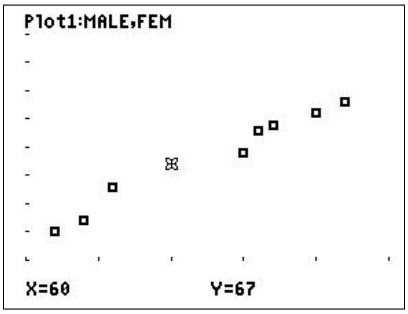
```
1-Var Stats
↑Sx=11.17546013
σx=10.65538315
n=11
minX=30
Q1=31
Med=36
Q3=43
maxX=62
```

Exercise page 109

The table shows the life expectancies of males and females in the period 1900 to 1990.

Year	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990
Males	52	54	56	60	65	66	67	67	70	72
Females	55	57	63	67	69	73	74	74	76	78

(a) Construct a scatterplot with female life expectancy on the vertical axis and male life expectancy on the horizontal axis.



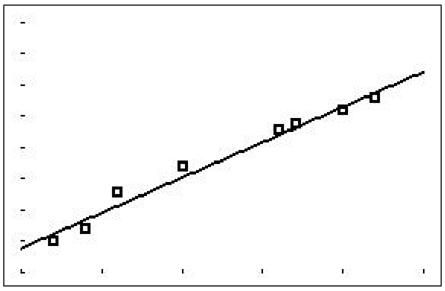
window $[50, 75, 5] \times [50, 90, 5]$

Determine

(i) the equation of the least-squares line for these data: $y = 1.13x - 2.5$.

(ii) The Pearson correlation coefficient r and the coefficient of determination r^2: $r = 0.98$, $r^2 = 0.97$.

(b) Plot the least-squares line on the scatterplot.



(c) Predict female life expectancy when the male life expectancy reaches 80.

From the equation of the liine of best fit, the female life expectancy when the male life expectancy reaches 80 is given by $y(80) = 1.13 \times 80 - 2.5 = 87.9$, i.e. 87.9 years.