TI in Focus: AP® Calculus

2020 Mock AP® Calculus Exam

AB-2: Solutions, Concepts, and Scoring Guidelines

Stephen Kokoska
Professor, Bloomsburg University
Former AP® Calculus Chief Reader

*AP is a registered trademark of the College Board, which was not involved in the production of, and does not endorse, this product.

AB 2

t	0	2	6	8	10	12
$y'(t)$	4	8	-2	3	-1	-5

The vertical position of a particle moving along the y-axis is modeled by a twice-differentiable function $y(t)$ where t is measured in seconds and $y(t)$ is measured in meters.

Selected values of $y'(t)$, the derivative of $y(t)$, over the interval $0 \leq t \leq 12$ seconds are shown in the table above.

The position of the particle at time $t = 12$ is $y(12) = -3$.

(e) Using a midpoint Riemann sum and three subintervals of equal length, approximate $\int_0^{12} y'(t)\,dt$.

Key Concepts

Midpoint Riemann Sum

Consider the continuous function f on the interval $[a, b]$.

Divide the interval $[a, b]$ into n nonoverlapping subintervals as defined by $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

This is called a **partition** of the interval $[a, b]$: $P = \{x_0, x_1, x_2, \ldots, x_n\}$

Let Δx_i be the length of the ith subinterval: $\Delta x_i = x_i - x_{i-1}$

Let x_i^* be an arbitrary value in the ith subinterval: $x_i^* \in [x_{i-1}, x_i]$

A **Riemann sum** of the function f over the interval $[a, b]$ with partition P:

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

Midpoint Riemann Sum

For a given partition P, a different Riemann sum is obtained for each collection of the x_i^*s.

(1) If x_i^* is the left endpoint of each subinterval: $x_i^* = x_{i-1}$.
This is a **left Riemann sum**.

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x_i$$

(2) If x_i^* is the right endpoint of each subinterval: $x_i^* = x_i$.
This is a **right Riemann sum**.

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

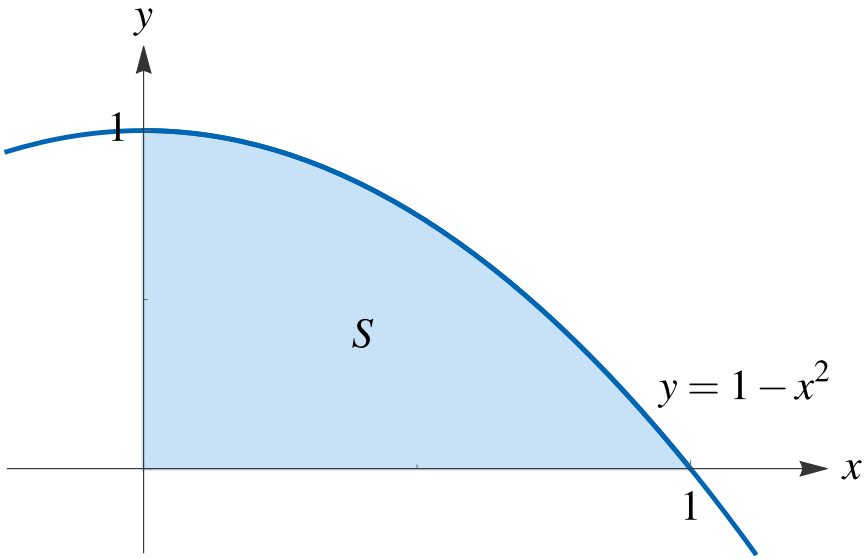
(3) If x_i^* is the midpoint of each subinterval: $x_i^* = \overline{x}_i = \frac{1}{2}(x_i + x_{i-1})$
This is called a **midpoint Riemann sum**.

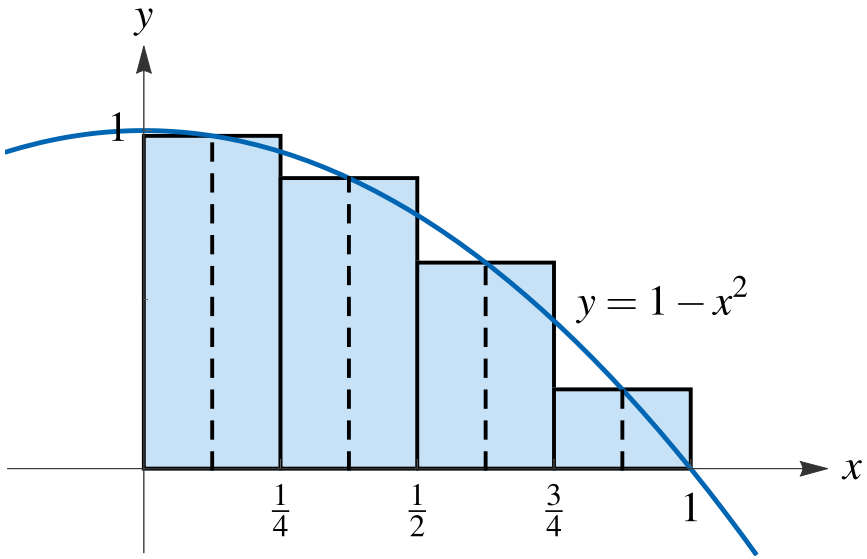
$$M_n = \sum_{i=1}^{n} f(\overline{x}_i)\,\Delta x_i$$

Example 1 Estimate the Area

Estimate the area of the region S bounded by the graph of $y = 1 - x^2$, the x-axis, and the y-axis.

Solution





$$\text{Area} \approx f\left(\frac{1}{8}\right)\cdot\frac{1}{4} + f\left(\frac{3}{8}\right)\cdot\frac{1}{4} + f\left(\frac{5}{8}\right)\cdot\frac{1}{4} + f\left(\frac{7}{8}\right)\cdot\frac{1}{4}$$

$$= \cdots = \frac{43}{64}$$

Definite Integral

If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ defined by

$$P = \{a = x_0, x_1, x_2, x_3, \ldots, x_n = b\}$$

Choose sample points x_i^* in $[x_{i-1}, x_i]$, the ith subinterval, and let $\Delta x_i = x_i - x_{i-1}$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i^*$$

provided the limit exists and gives the same value for all possible choices of sample points.

If the limit does exists, we say that f is **integrable** on $[a, b]$.

A Closer Look

1. The precise meaning of the limit:
   For every number $\epsilon > 0$ there exists an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i \right| < \epsilon$$

   for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

2. The norm of a partition P: the maximum width of all subintervals, denoted $||P||$.
   Therefore, $||P|| = \max\{\Delta x_i\}$. As $n \to \infty$, $||P|| \to 0$

$$\int_a^b f(x)\,dx = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

3. Interpretation: The definite integral is the limit of a Riemann sum and an integrable function can be approximated by a Riemann sum.

Solution

Three subintervals of equal length: $P = \{0, 4, 8, 12\}$

Midpoint of each subinterval: $x_1^* = 2, \quad x_2^* = 6, \quad x_3^* = 10$

$$\int_0^{12} y'(t)\,dt \approx y'(x_1^*)\,\Delta x_1 + y'(x_2^*)\,\Delta x_2 + y'(x_3^*)\,\Delta x_3$$

$$= y'(2)\cdot(4-0) + y'(6)\cdot(8-4) + y'(10)\cdot(12-8)$$

$$= (8)(4) + (-2)(4) + (-1)(4) = 20$$

Scoring Guidelines

(e) $\displaystyle\int_0^{12} y'(t)\,dt$

$$\approx y'(2)\cdot(4-0) + y'(6)\cdot(8-4) + y'(10)\cdot(12-8)$$

$$= (8)(4) + (-2)(4) + (-1)(4) = 20$$

$$2: \begin{cases} 1 : \text{midpoint Riemann sum} \\ 1 : \text{estimate} \end{cases}$$

Scoring Notes

1: midpoint Riemann sum

- Earned for a correct midpoint sum: a method point.
- Must see correct multiplication and evidence of a sum.
- The sum can have one error and still earn this point: 5/6 rule. (Not eligible for the second point.)

1: estimate

- A bald answer does not earn any points.
- The numerical values for $y'(2)$, $y'(6)$, and $y'(10)$ must be pulled from the table.
- Any simplification must be correct.

(f) Using correct units, explain the meaning of $\int_0^{12} y'(t)\,dt$ in the context of the problem.

Key Concepts

Displacement

Suppose a particle moves along a horizontal line so that its position at time t is given by $s(t)$, then its velocity is $v(t) = s'(t)$, and

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change in position, or **displacement**, of the particle over the time period from t_1 to t_2.

Interpretation: the net distance traveled, by the particle over the interval $[t_1, t_2]$.

Solution

$y(t)$ is the vertical position of the particle at time t.

$y'(t)$ is the velocity of the particle at time t.

$$\int_0^{12} y'(t)\,dt$$

The (vertical) displacement, or the change in position, or the net distance traveled, in meters, by the particle over the time interval $[0, 12]$ seconds.

Scoring Guidelines

(f) $\int_0^{12} y'(t)\,dt$ is the vertical displacement (change in position), in meters, of the particle over the time interval $0 \le t \le 12$ seconds.	$2: \begin{cases} 1 : \text{interpretation} \\ 1 : \text{units} \end{cases}$

Scoring Notes

1: interpretation

- Interpretation must clearly indicate the time interval $t = 0$ to $t = 12$.
  It cannot reference a specific point in time (at time $t = 12$, or after $t = 12$ seconds).
- Some acceptable ways of indicating the time interval:
  On the time interval $[0, 12]$; $0 \leq t \leq 12$; Over the first 12 seconds;
  During the first 12 seconds; During the 12 second time interval.
- Any unit of time can be used to earn the first point.
  Incorrect time units: will not earn the second point.
- Interpretation must include: displacement, or change in position, or net distance traveled.

1: units

- Units must include the distance units (meters) but do not need to include the t (time) units.
  However, if the time units are presented, they must be correct.

education.ti.com