TI in Focus: AP® Calculus

2019 AP® Calculus Exam: AB-1/BC-1
Average Value of a Function

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Outline

(1) Background; motivation

(2) Derivation

(3) Definition and interpretation

(4) Examples

(5) Mean Value Theorem for Integrals

Introduction

(1) There are often questions involving average value of a function on the AP Calculus Exam.

(2) Objective: Visualize and compute the average value of a function.

(3) Prerequsites:

   (a) Antidifferentiation rules and techniques

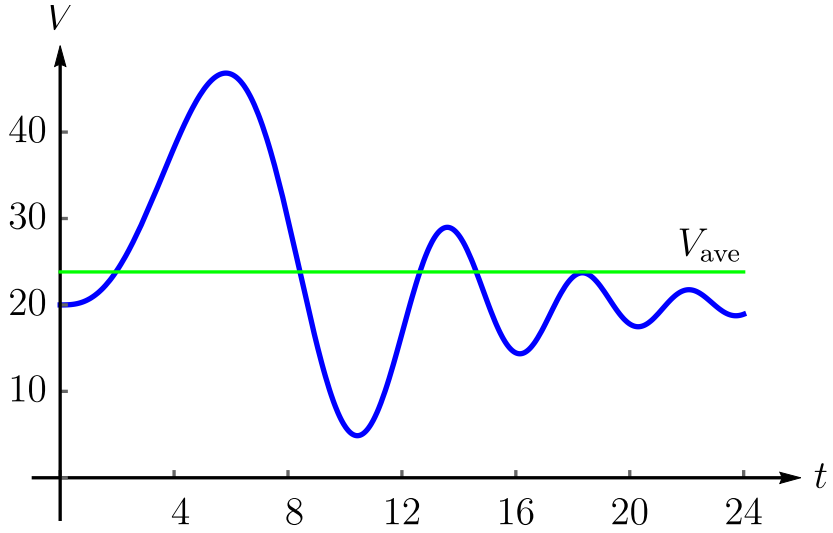
   (b) The Fundamental Theorem of Calculus

Background

- Consider the arithmetic mean of a set of numbers, $x_1, x_2, \ldots, x_n$

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

- How do we compute the *average* volume of water in a reservoir over a time interval? Infinitely many volume measurements possible.
- Illustration: Graph of $V(t)$, where t is measured in hours and V in billions of gallons. V_{ave} is a guess at the average volume.



Objective: Compute the average value of a function $y = f(x)$ over the interval $a \leq x \leq b$.

- Construct a regular partition of $[a, b]$:

  Divide $[a, b]$ into n equal subintervals, each of width $\Delta x = \dfrac{b-a}{n}$

- Select x_i^* in the ith subinterval.

  $x_i^* \in [x_{i-1}, x_i]$

- Compute the average: $\dfrac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$

- $\Delta x = \dfrac{b-a}{n} \implies n = \dfrac{b-a}{\Delta x}$

- Rewrite the average value.

The average value:

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\frac{b-a}{\Delta x}}$$

$$= \frac{1}{b-a}\left[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\right]\Delta x$$ Rewrite fraction

$$= \frac{1}{b-a}\left[f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$ Distribute Δx

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$ Use summation notation

This summation is a Riemann sum.
The limiting value is the definition of a definite integral.

The Riemann sum:

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Definition: The average value of f on the interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

If $f(x) \geq 0$, here is an interpretation:

$$\text{average height} = \frac{\displaystyle\int_a^b f(x)\,dx}{b-a} = \frac{\text{area}}{\text{width}}$$

Example 1 Average Value of a Function

Find the average value of the function $f(x) = 3 + 2x - x^2$ on the interval $[0, 3]$.

Solution

The interval is $[0, 3]$: $a = 0$ and $b = 3$

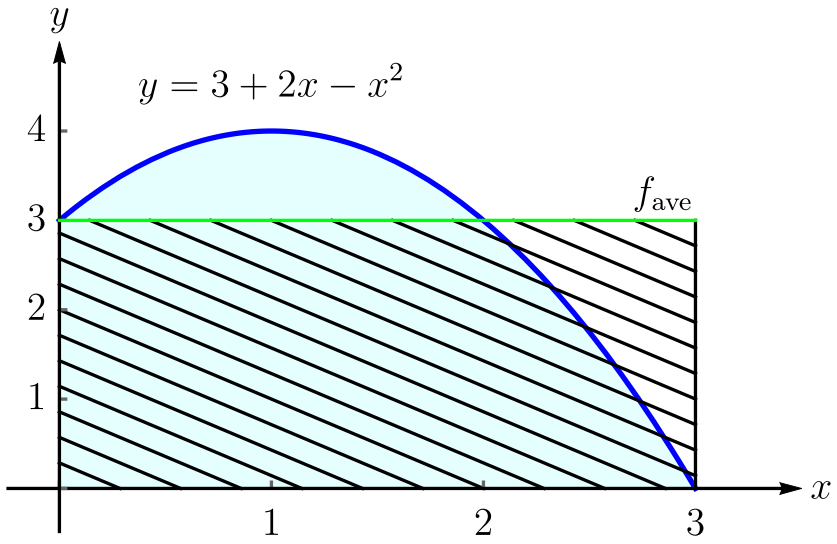
$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3-0}\int_0^3 (3 + 2x - x^2)\,dx$$ Definition of average value

$$= \frac{1}{3}\left[3x + x^2 - \frac{x^3}{3}\right]_0^3$$ Basic antidifferentiation rules

$$= \frac{1}{3}[(9 + 9 - 9) - (0)] = 3$$ FTC; simplify

Solution (Continued)

The graph of f and the line $y = f_{\text{ave}}$:



The area of the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is the same as the area of the rectangle with height f_{ave} and width $b - a$.

Example 2 Average Value Involving a Trigonometric Function

The function f is given by $f(x) = 2 + \cos x$.

(a) Find the average value of the function f over the interval $\left[-\frac{\pi}{6}, \frac{5\pi}{6}\right]$.

(b) Carefully sketch a graph to illustrate this result.

Background

(1) It seems reasonable to ask if there is a point when the value of the function is equal to its average value.

(2) For the reservoir volume example, there are at least four times.

(3) For a continuous function, there is at least one number c at which the value of the function is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$.

The Mean Value Theorem for Integrals

If f is a continuous function on $[a, b]$, then there exists a number c in $[a, b]$ such that

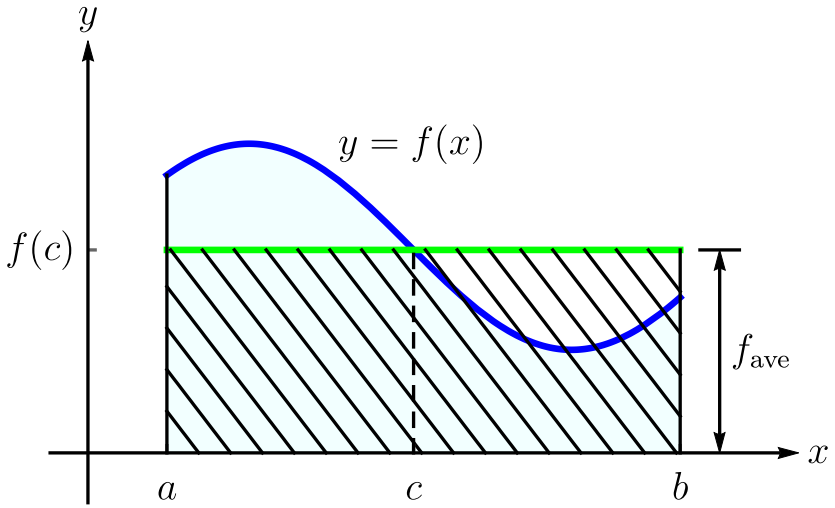
$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

That is, $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a)$

A Closer Look

(1) The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for Derivatives and the Fundamental Theorem of Calculus. Proof?

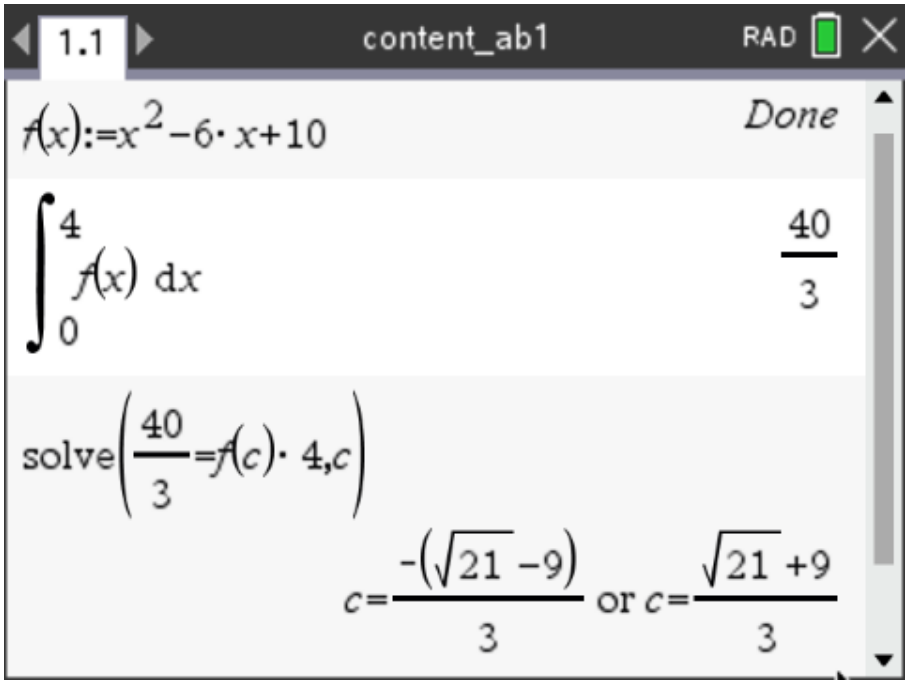
(2) Geometric illustration:

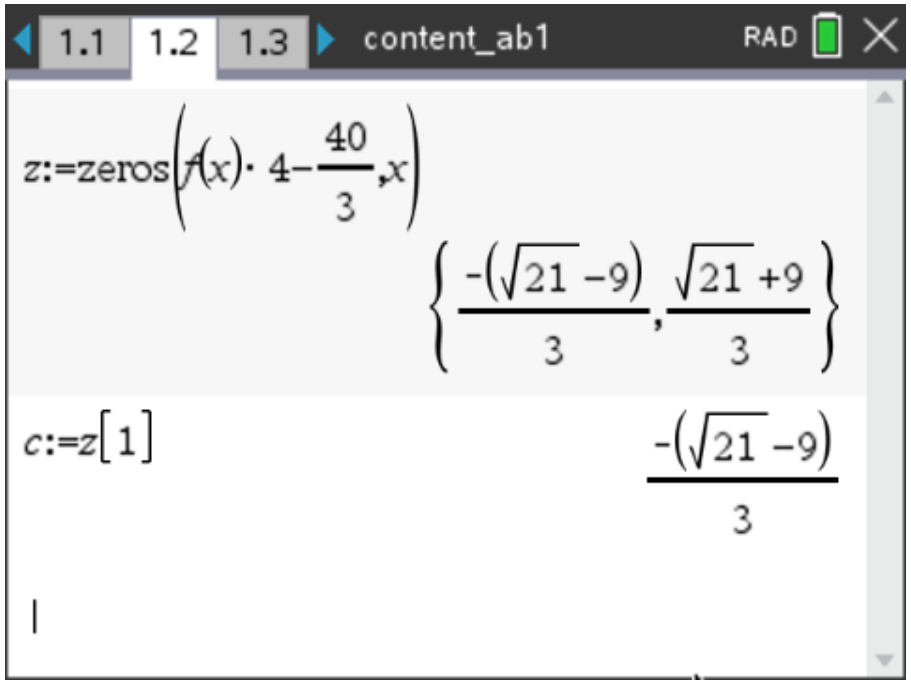


Example 3 MVT for Integrals

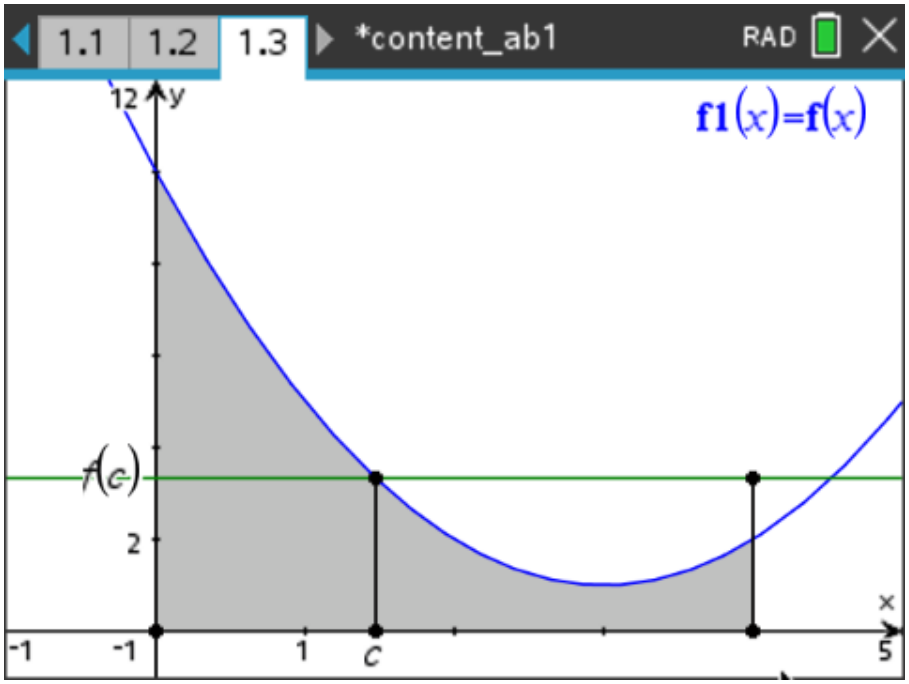
The function f is given by $f(x) = x^2 - 6x + 10$. f is continuous on $[0, 4]$. Therefore, the MVT for Integrals says there is a number c in $[0, 4]$ such that

$$\int_0^4 (x^2 - 6x + 10)\, dx = f(c)[4 - 0]$$





Here is a graph that illustrates this conclusion.



Example 4 Tea Time

A cup of tea with temperature $180°$F is placed in a room with temperature $70°$F. The temperature of the tea after t minutes is given by

$$T(t) = 70 + 110e^{-0.025t}$$

(a) What is the average temperature of the tea during the first half hour?

(b) Carefully sketch a graph to illustrate this conclusion.

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