TI in Focus: AP® Calculus

2018 AP® Calculus Exam: BC-5
Scoring Guidelines

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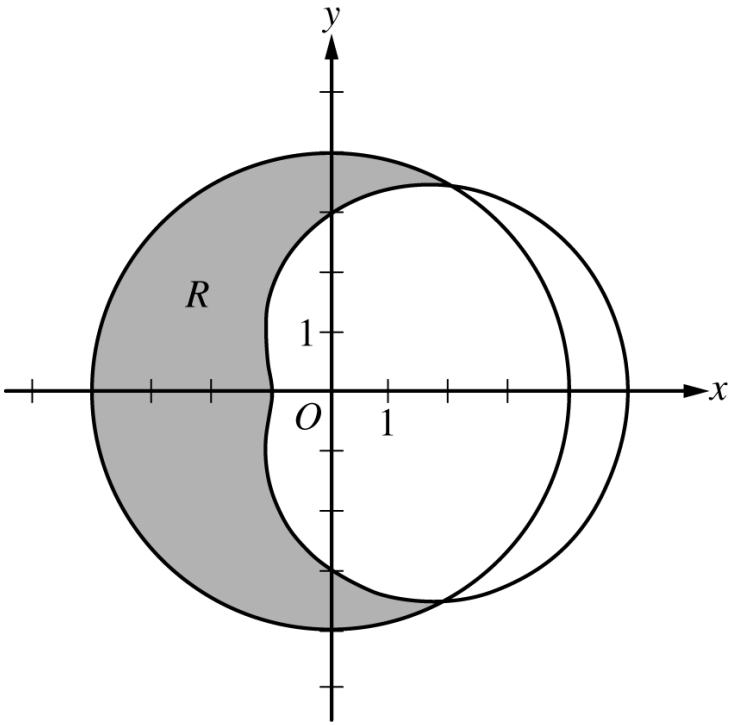

5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

(b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

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(c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

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(a) $\text{Area} = \frac{1}{2}\int_{\pi/3}^{5\pi/3}\left(4^2 - (3 + 2\cos\theta)^2\right)d\theta$

$3 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \end{cases}$

(b) $\frac{dr}{d\theta} = -2\sin\theta \Rightarrow \left.\frac{dr}{d\theta}\right|_{\theta=\pi/2} = -2$

$r\left(\frac{\pi}{2}\right) = 3 + 2\cos\left(\frac{\pi}{2}\right) = 3$

$y = r\sin\theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

$x = r\cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{-2\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

$3 : \begin{cases} 1 : \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \\ \quad \text{or } \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ 1 : \frac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ 1 : \text{answer} \end{cases}$

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Part (b) Alternate Solution

$$y = r\sin\theta = (3 + 2\cos\theta)\sin\theta \Rightarrow \frac{dy}{d\theta} = 3\cos\theta + 2\cos^2\theta - 2\sin^2\theta$$

$$x = r\cos\theta = (3 + 2\cos\theta)\cos\theta \Rightarrow \frac{dx}{d\theta} = -3\sin\theta - 4\sin\theta\cos\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{2}\right) - 2\sin^2\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

(c) $$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = -2\sin\theta\cdot\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{dr}{dt}\cdot\frac{1}{-2\sin\theta}$$

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = 3\cdot\frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \text{ radians per second}$$

$$3 : \begin{cases} 1 : \frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} \\ 1 : \frac{d\theta}{dt} = \frac{dr}{dt}\cdot\frac{1}{-2\sin\theta} \\ 1 : \text{answer with units} \end{cases}$$

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Student Performance

Part (a)

- Intersection given: Most students earned constants and limits point.
- Use of the entire of the area of the outer circle, 16π rather than $\dfrac{32\pi}{3}$.
- Most students were able to write the correct integrand.
- Common errors: integrand reversal, square of a difference of functions, adding the squares, failing to square anything.

Part(b)

- Most students tried to solve this problem using the second solution on the scoring guidelines.
- Common errors: algebra errors in computing x and y, sign errors in derivatives, writing, organizing, arranging $\dfrac{dy}{dx}$.
- Most students used the Product Rule and the Chain Rule successfully.
- Some students used $\dfrac{dr}{d\theta}$ as the slope of the tangent line.

Student Performance

Part (c)

- Most students recognized this as a related rates problem.
- Most students used the Chain Rule to find $\frac{dr}{dt}$ and then $\frac{d\theta}{dt}$.
- Common errors: incorrect $\frac{dr}{d\theta}$,
  algebra or arithmetic errors in isolating $\frac{d\theta}{dt}$,
  trigonometric errors in evaluating $\sin\frac{\pi}{3}$,
  unit mistakes (degrees per second).

Part (a) 1: constant and limits

- Need $\frac{1}{2}$ and the limits of $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.
- However, could use a doubling argument.
- Any integrand is OK (even if $d\theta$).
- 16π does not earn the point. $\frac{32\pi}{3}$ is eligible.

Part (a) 2: integrand

- Eligibility: integrand is difference of two squared terms (special case).
- r or $r(\theta)$ not sufficient.
- Ignore subsequent algebra mistakes unless a final answer with these mistakes is boxed.
- Extra integral or term that produces an incorrect answer: ? - 1
- Reversal without the limits reversed: ? - 1

Part (a) 2: integrand, Special Case

Earns 1/2 if

- Constant and limits point is earned, and
- Both squared expressions are correct, and
- Two square expressions appear in the correct order: $4^2 + (3 + 2\cos\theta)^2$

Part (a) Examples

(1) $\frac{1}{2}\int_{\pi/3}^{5\pi/3} 4^2\,d\theta - \frac{1}{2}\int_{\pi/3}^{5\pi/3} (3+2\cos\theta)^2\,d\theta$ 1 - 2

(2) $\frac{32\pi}{3} - \frac{1}{2}\int_{\pi/3}^{5\pi/3} (3+2\cos\theta)^2\,d\theta$ 1 - 2

(3) $16\pi - \frac{1}{2}\int_{\pi/3}^{5\pi/3} (3+2\cos\theta)^2\,d\theta$ 0 - 2

(4) $\frac{1}{2}\int_{\pi/3}^{5\pi/3} [4-(3+2\cos\theta)]^2\,d\theta$ 1 - 0

(5) $\frac{1}{2}\int_{\pi/3}^{5\pi/3} [(3+2\cos\theta)^2 - 4^2]\,d\theta$ 1 - 1

(6) $\frac{1}{2}\int_{\pi/3}^{5\pi/3} [(3+2\cos\theta)^2 + 4^2]\,d\theta$ 1 - 0

Part (b) 1: $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + 4\cos\theta$ or $\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

- Need one correct derivative with respect to θ.
- Just dy and just dx are OK here.
- May substitute $r = 3 + 2\cos\theta$ first.
- Read part (b) work in part (c) only if directed to.
- Many correct forms of x and y using $r = 3 + 2\cos\theta$.
  (First point not earned yet)

$$x = (3 + 2\cos\theta)\cos\theta$$
$$x = 3\cos\theta + 2\cos^2\theta$$

$$y = (3 + 2\cos\theta)\sin\theta$$
$$y = 3\sin\theta + 2\cos\theta\sin\theta$$
$$y = 3\sin\theta + \sin(2\theta)$$

Part (b) Correct forms of $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$

Any one of these earns the first point.

- $\dfrac{dx}{d\theta} = (-2\sin\theta)(\cos\theta) + (3 + 2\cos\theta)(-\sin\theta)$
- $\dfrac{dx}{d\theta} = -3\sin\theta - 4\sin\theta\cos\theta$
- $\dfrac{dx}{d\theta} = -3\sin\theta - 2\sin(2\theta)$
- $\dfrac{dy}{d\theta} = (-2\sin\theta)(\sin\theta) + (3 + 2\cos\theta)(\cos\theta)$
- $\dfrac{dy}{d\theta} = 3\cos\theta + 2\cos^2\theta - 2\sin^2\theta$
- $\dfrac{dy}{d\theta} = 3\cos\theta + 2\cos(2\theta)$

Part (b) 1: $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

- The expression above (on the scoring guidelines) earns the first two points.
- Do not need to substitute $r = 3 + 2\cos\theta$.
- Do not need to substitute $\theta = \pi/2$ yet.

Philosophy: Use $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ correctly or consistently to assemble $\frac{dy}{dx}$.

- $\frac{dy}{d\theta}$ **or** $\frac{dx}{d\theta}$ incorrect, **and** $\frac{dy}{dx}$ consistent. 1 - 1 - 0
- $\frac{dy}{d\theta}$ **and** $\frac{dx}{d\theta}$ incorrect (not overly simplified), **and** $\frac{dy}{dx}$ consistent. 0 - 1 - 0
- $\frac{dy}{d\theta}$ **or** $\frac{dx}{d\theta}$ incorrect, **and** $\frac{dy}{dx}$ inconsistent. 1 - 0 - 0
- $\frac{dy}{d\theta}$ **and** $\frac{dx}{d\theta}$ incorrect, **and** $\frac{dy}{dx}$ inconsistent. 0 - 0 - 0

Part (b) 1: answer

- Must be our answer.
- Declared correct slope inserted into correct tangent line: earns the point.
- Declared correct slope inserted into incorrect tangent line earns the point, **unless** final incorrect tangent line is boxed.
- Undeclared correct slope inserted into correct tangent line: earns the point.
- Undeclared correct slope inserted into incorrect tangent line does not earn the point.

Examples

(1) $$\frac{dy}{dx} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos\left(2\left(\frac{\pi}{2}\right)\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)}$$ 1 - 1 - 1

(2) $$\frac{dy}{dx} = \frac{-2\sin\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right)}$$? - ? - 1

Part (c) 1: $\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$

- Point is earned for the formula with additional work in this section.
- If earned, the point is in the bank.
- If the expression appears in part (b): additional work in part (c): 1 - ? - ?
- If the expression appears in part (b): no additional work in part (c): 0 - 0 - 0
- No $\frac{d\theta}{dt}$: 0 - 0 - 0

Examples

(1) $\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$? - ? - ?

(2) $\frac{dr}{dt} = -2\sin\theta \cdot \frac{d\theta}{dt}$ 1 - ? - ?

(3) $\frac{dr}{dt} = 2\sin\theta \cdot \frac{d\theta}{dt}$ 0 - ? - ?

Part (c) 1: $\frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin\theta}$

- $\dfrac{d\theta}{dt}$ isolated in an equation.
- May isolate this term symbolically or numerically.
- If there is an algebra error in isolating this term: point is not earned.
- If $\dfrac{dr}{d\theta}$ is incorrect, check for the first appearance.

Examples

(1) $\dfrac{dr}{dt} = -2\sin\theta \cdot \dfrac{d\theta}{dt} \implies \dfrac{d\theta}{dt} = \dfrac{dr}{dt} \cdot \dfrac{1}{-2\sin\theta}$ 1 - 1 - ?

(2) $\dfrac{dr}{dt} = \dfrac{dr}{d\theta} \cdot \dfrac{d\theta}{dt} \implies \dfrac{d\theta}{dt} = \dfrac{dr}{dt} \cdot \dfrac{1}{2\sin\theta}$ 1 - 0 - ?

(3) $\dfrac{dr}{dt} = 2\sin\theta \cdot \dfrac{d\theta}{dt} \implies \dfrac{d\theta}{dt} = \dfrac{dr}{dt} \cdot \dfrac{1}{2\sin\theta}$ 0 - 1 - ?

Part (c) 1: answer with units

- Read for the units first (radians per second).
- Common errors: degrees per second, units per second.
- The final numerical answer does not need to be simplified.
- If there is a trigonometric error in evaluating $\frac{d\theta}{dt}$: point not earned.
- Incorrect $\frac{dr}{d\theta}$ and consistent answer with units: point earned.
- Bald answer: 0 - 0 - 0

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