

## TI in Focus: AP<sup>®</sup> Calculus

2018 AP<sup>®</sup> Calculus Exam: BC-2

Arc Length and Parametric Equations

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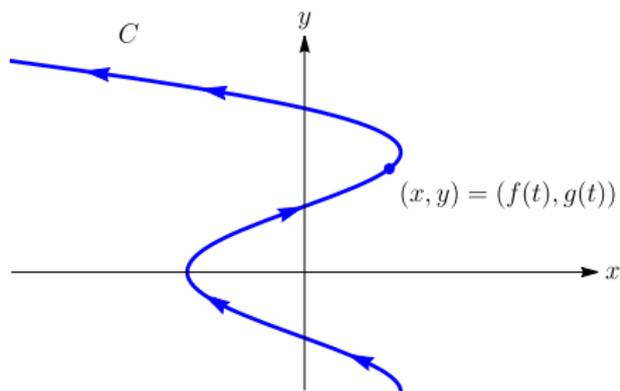
Former AP<sup>®</sup> Calculus Chief Reader

## Outline

- (1) Background: parametric equations
- (2) Length of a path
- (3) Polygonal path approximations
- (4) Length of a curve  $C$
- (5) Particle motion
- (6) Examples

## Background

Suppose a particle moves along the curve  $C$  as shown.



- Cannot describe  $C$  by an equation of the form  $y = f(x)$ .
- Consider the  $x$ - and  $y$ -coordinates of the particle as functions of a third variable  $t$ , often corresponding to time.
- Write:  $x = f(t)$  and  $y = g(t)$ .

## Definition

Suppose  $x$  and  $y$  are both given as functions of a third variable  $t$ , called a **parameter**, by

$$x = f(t) \quad y = g(t)$$

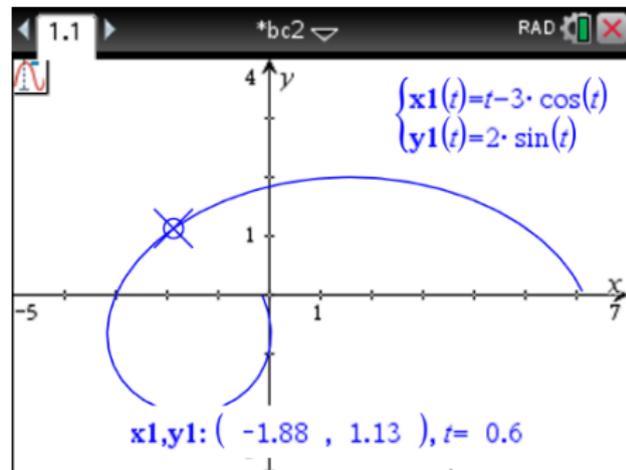
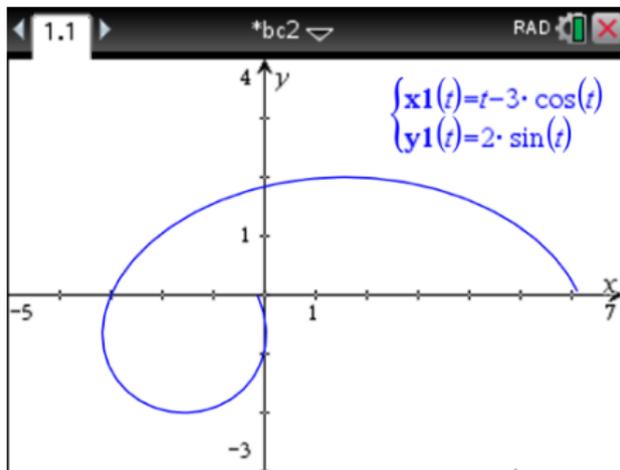
called **parametric equations**.

- Each value of  $t$  determines a point  $(x, y)$ , which we can plot in the coordinate plane.
- As  $t$  varies, the point  $(x, y) = (f(t), g(t))$  varies and traces out a curve  $C$ , a **parametric curve**.
- If  $t$  represents time:  $(x, y) = (f(t), g(t))$  is the position of the particle at time  $t$ .

## Example 1 Curve Sketching

Sketch the curve defined by the parametric equations  $x = t - 3 \cos t$ ,  $y = 2 \sin t$ ,  $-\pi \leq t \leq \pi$ .

### Solution



## Example 2 Short Walk

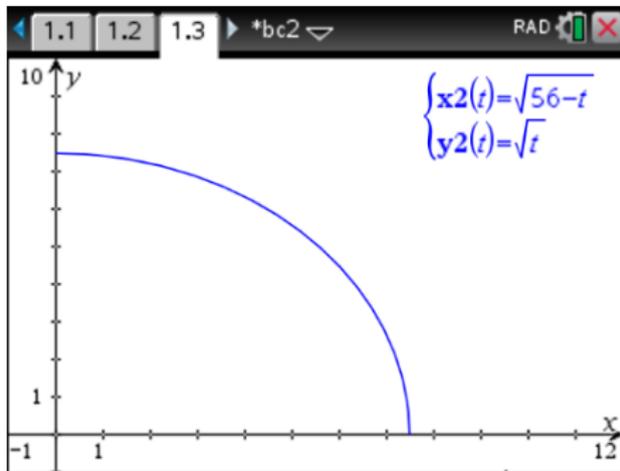
A particle moves about in the plane so that its position at time  $t$  is given by the parametric equations

$$x = \sqrt{56 - t} \quad y = \sqrt{t}$$

Find the length of the path of the particle.

### Solution

Domain:  $0 \leq t \leq 56$



## Solution

Eliminate the variable  $t$

$$x^2 + y^2 = [\sqrt{56-t}]^2 + [\sqrt{t}]^2 = (56-t) + t = 56$$

The path of the particle is part of circle centered at the origin,  $r = \sqrt{56}$ .

$$\text{Length of the path} = \frac{1}{4}2\pi r = \frac{1}{4}2\pi \cdot \sqrt{56} = \frac{\pi\sqrt{56}}{2} = \pi\sqrt{14}$$

## Arc Length

- If  $F'(x)$  is continuous on  $[a, b]$  then the length of the curve defined by  $y = F(x)$ ,  $a \leq x \leq b$  is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

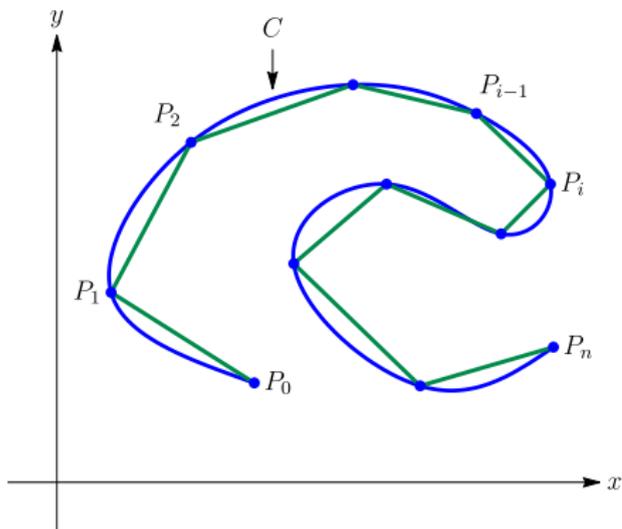
- Suppose  $C$  is described by  $x = f(t)$ ,  $y = g(t)$ ,  $\alpha \leq t \leq \beta$ ,  $\frac{dx}{dt} = f'(t) > 0$

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt} dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \end{aligned}$$

$$dx = f'(t) dt$$

## Polygonal Paths

- Divide the interval  $[\alpha, \beta]$  into  $n$  subintervals of equal width  $\Delta t$ .
- Endpoints of the subintervals:  $t_0, t_1, t_2, \dots, t_n$
- $x = f(t_i)$  and  $y = g(t_i)$  are the coordinates of points  $P_i(x_i, y_i)$  that lie on  $C$ .
- Polygonal path with vertices  $P_0, P_1, P_2, \dots, P_n$  approximates  $C$ .



## Polygonal Paths to the Limit

- Length  $L$  of  $C$ : 
$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n |P_{i-1}P_i|$$

- Apply the MVT to  $f$  on  $[t_{i-1}, t_i]$ .

There exists a number  $t_i^*$  in  $(t_{i-1}, t_i)$  such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

- Let  $\Delta x_i = x_i - x_{i-1}$ ,  $\Delta y_i = y_i - y_{i-1}$ :  $\Delta x_i = f'(t_i^*) \Delta t$

- Apply the MVT to  $g$  on  $[t_{i-1}, t_i]$ .

There exists a number  $t_i^{**}$  in  $(t_{i-1}, t_i)$  such that

$$\Delta y_i = g'(t_i^{**}) \Delta t$$

## Polygonal Paths to the Limit

- Length of each straight-line segment:

$$\begin{aligned}|P_{i-1}P_i| &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*) \Delta t]^2 + [g'(t_i^{**}) \Delta t]^2} \\ &= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2} \Delta t\end{aligned}$$

- Length of the curve:

$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2} \Delta t$$

- This is *almost* a Riemann sum  $t_i^* \neq t_i^{**}$  in general. But...

- $$\begin{aligned}L &= \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt \\ &= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt\end{aligned}$$

## Theorem

If a curve  $C$  is described by the parametric equations  $x = f(t)$ ,  $y = g(t)$ ,  $\alpha \leq t \leq \beta$ , where  $f'$  and  $g'$  are continuous on  $[\alpha, \beta]$  and  $C$  is traversed exactly once as  $t$  increases from  $\alpha$  to  $\beta$ , then the length of the  $C$  is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

## Particle Motion

Suppose a particle moves along a curve so that its position vector at time  $t$  is  $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ . The velocity vector  $\mathbf{v}(t)$  is given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

Total distance traveled by the particle from time  $t = \alpha$  to time  $t = \beta$ :

$$\text{distance traveled} = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### Example 3 Arc Length

Graph the curve described by the parametric equations and find its length.

(a)  $x = (2\pi - t)e^{t/2} \cos t$ ,  $y = e^{t/2} \sin t$ ,  $0 \leq t \leq 2\pi$

(b)  $x = \ln t + \sin t$ ,  $y = 10e^{-t} \cos t$ ,  $1 \leq t \leq 6$

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