

TI in Focus: AP[®] Calculus

2017 AP[®] Calculus Exam: BC-2

Areas in Polar Coordinates

Stephen Kokoska

Professor, Bloomsburg University

Former AP[®] Calculus Chief Reader

Outline

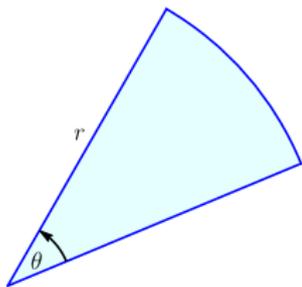
- (1) Derivation of the area formula
- (2) Example
- (3) Area between two curves
- (4) Another example

Background

(1) Objective: develop a formula for the area of a region bounded by the graph of a polar equation.

(2) Recall: area of a sector of a circle: $A = \frac{1}{2}r^2\theta$

r : radius of the circle; θ : radian measure of the central angle

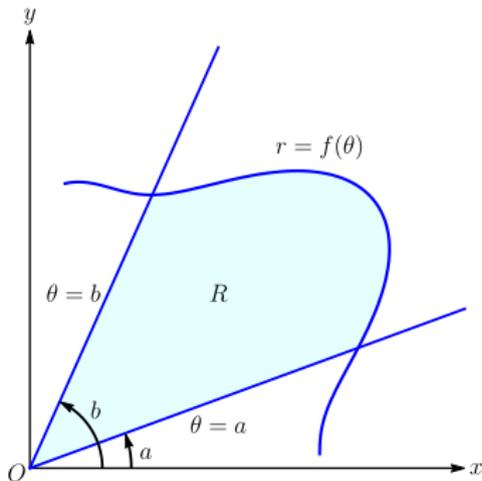


Area of a sector is proportional to its central angle:

$$A = \left(\frac{\theta}{2\pi}\right) \pi r^2 = \frac{1}{2}r^2\theta$$

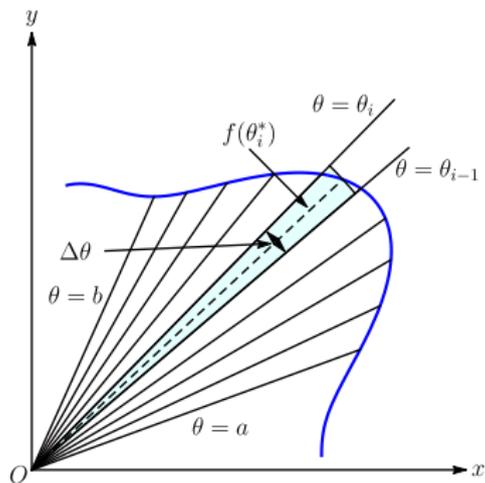
Area of a Polar Region

- Let R be the region bounded by the graph of the polar equation $r = f(\theta)$ and the rays $\theta = a$, and $\theta = b$.
- Assume f is a positive continuous function and $0 < b - a \leq 2\pi$.



Area Approximation

- Divide the interval $[a, b]$ into n equal subintervals of width $\Delta\theta$.
- Endpoints of the interval:
 $\theta_0, \theta_1, \theta_2, \dots, \theta_n$.
- The rays $\theta = \theta_i$ divide the region R into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$.
- Choose θ_i^* in the i th subinterval $[\theta_{i-1}, \theta_i]$.
- Area of the i th region: ΔA_i is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$.



Riemann Sum

- Use the formula for the area of a sector of a circle:

$$\Delta A_i \approx \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta$$

- An approximation for the total area A of the region R :

$$A \approx \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta$$

- This approximation is a Riemann sum. Therefore,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$$

- The area A of the polar region R :

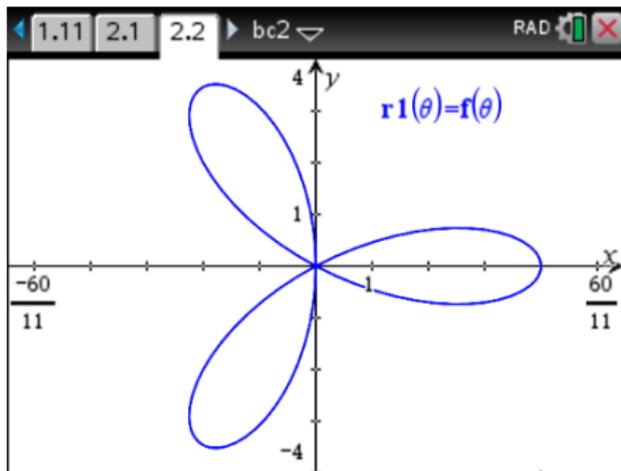
$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta$$

Example 1 Area of a Petal

Find the area enclosed by one loop of the polar curve described by $r = 4 \cos(3\theta)$.

Solution

The top half of the loop in Quadrant I and IV is traced out for $0 \leq \theta \leq \frac{\pi}{6}$.



Solution

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/6} [4 \cos(3\theta)]^2 d\theta$$

Expression for r

$$= \int_0^{\pi/6} 16 \cos^2 3\theta d\theta = 16 \int_0^{\pi/6} \frac{1}{2}(1 + \cos 6\theta) d\theta$$

Square; half-angle formula

$$= 8 \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6}$$

Antiderivatives

$$= 8 \left[\left(\frac{\pi}{6} + \frac{1}{6} \sin \pi \right) - (0) \right]$$

Evaluate

$$= \frac{8\pi}{6} = \frac{4\pi}{3}$$

Simplify

Solution

The image shows a TI-84 Plus calculator screen. At the top, the mode is set to RAD. The function $f(\theta) := 4 \cdot \cos(3 \cdot \theta)$ is defined. Below this, the integral $2 \cdot \frac{1}{2} \cdot \int_0^{\frac{\pi}{6}} (f(\theta))^2 d\theta$ is calculated, resulting in $\frac{4 \cdot \pi}{3}$. The word "Done" is visible in the top right corner of the screen.

2.1 2.2 2.3 *bc2 ▾ RAD

$f(\theta) := 4 \cdot \cos(3 \cdot \theta)$ Done

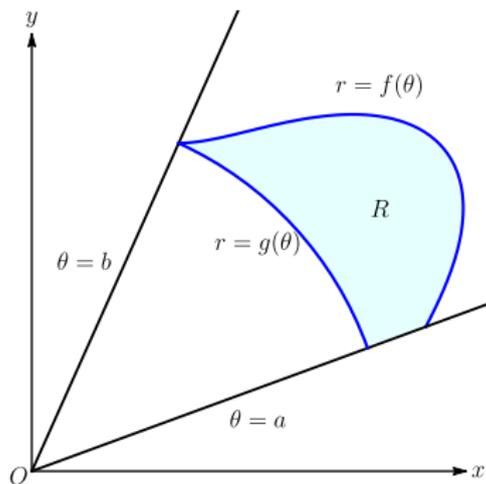
$2 \cdot \frac{1}{2} \cdot \int_0^{\frac{\pi}{6}} (f(\theta))^2 d\theta$ $\frac{4 \cdot \pi}{3}$

Area of a region bounded by two polar curves

- Let R be a region bounded by the curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$.
- The area A of R is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$.

$$\begin{aligned}
 A &= \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta - \int_a^b \frac{1}{2} [g(\theta)]^2 d\theta \\
 &= \frac{1}{2} \int_a^b ([f(\theta)]^2 - [g(\theta)]^2) d\theta
 \end{aligned}$$

Note: Because a single point has many representations in polar coordinates, need to carefully consider the points of intersection of two polar curves.

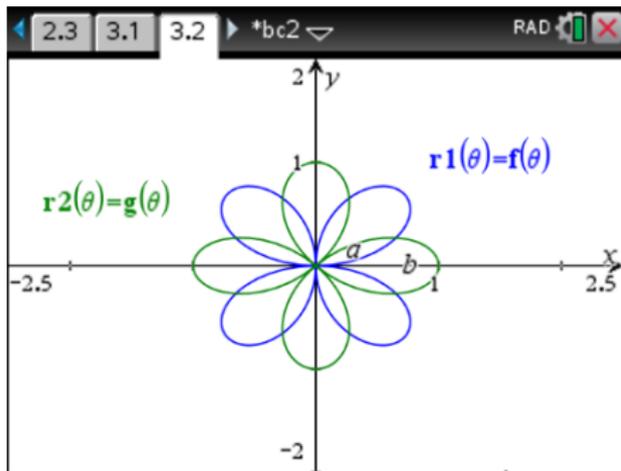


Example 2 Look Inside

Consider the polar curves described by $r = f(\theta) = \sin 2\theta$ and $r = g(\theta) = \cos 2\theta$.

- Find the area of the region that lies inside both curves.
- Find the area of the region that lies outside the graph of $r = f(\theta)$ and inside the graph of $r = g(\theta)$.

Solution



Solution

$$(a) A = 8 \cdot 2 \cdot \frac{1}{2} \int_0^{\pi/8} \sin^2 2\theta \, d\theta = \frac{\pi}{2} - 1$$

$$(b) A = 4 \cdot 2 \cdot \frac{1}{2} \int_0^{\pi/8} [\cos^2 2\theta - \sin^2 2\theta] \, d\theta = 1$$

The screenshot shows a TI-84 Plus calculator interface with the mode set to RAD. The top navigation bar shows tabs for 3.1, 3.2, and 3.3, with 3.3 selected. The input field contains '*bc2'. The display shows two integration problems:

Problem 1: $8 \cdot \int_0^{\frac{\pi}{8}} (\sin(2 \cdot \theta))^2 \, d\theta$ with the result $\frac{\pi-2}{2}$.

Problem 2: $4 \cdot \int_0^{\frac{\pi}{8}} ((\cos(2 \cdot \theta))^2 - (\sin(2 \cdot \theta))^2) \, d\theta$ with the result 1.

education.ti.com