2017 AP Calculus Exam: AB-6
The Chain Rule

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Background

(1) Differentiation rules: sum, difference, product, and quotient.

(2) Suppose F is a function defined by $F(x) = (x^3 - 7x^2 + 3)^{10}$.

(3) Expand $f(x)$; polynomial of degree 30; derivative term-by-term.

Note

(1) F is a composite function.

(2) Let $y = f(u) = u^{10}$ and $u = g(x) = x^3 - 7x^2 + 3$.

Then $y = F(x) = f(g(x))$

That is $F = f \circ g$

Intuition and Interpretation

(1) We can differentiate f; and differentiate g;

It seems reasonable that we can find a rule to differentiate $F = f \circ g$;

In terms of f and g.

(2) The derivative of a composite function $f \circ g$ is the product of the derivatives of f and g.

(3) One of the most important differentiation rules: *Chain Rule*.

(4) An interpretation in terms of rates of change:

$\dfrac{du}{dx}$: the rate of change of u with respect to x.

$\dfrac{dy}{du}$: the rate of change of y with respect to u.

If u changes twice as fast as x, and y changes three times as fast as u,

Then it seems reasonable that y changes six times as fast as x.

Therefore, we expect $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and $F'(x)$ is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Liebniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

A Closer Look

(1) In words: the derivative of the composite of two functions is the product of the derivative of the *outer* function evaluated at the *inner* function and of the derivative of the inner function.

Calculus vision: identify the inner and the outer functions.

(2) Other notation:

- $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
- $\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

(3) A procedure to find $(f \circ g)'(x)$ for a given $(f \circ g)(x)$.

Step 1: Identify $f(x)$ and $g(x)$.

Step 2: Find $f'(x)$ and $g'(x)$.

Step 3: Write the final answer as $f'(g(x))g'(x)$.

Example 1

Find $F'(x)$ if $F(x) = (x^3 - 7x^2 + 3)^{10}$.

Solution

Identify the *inner* and the *outer* functions.

$F(x) = (f \circ g)(x)$ where $f(x) = x^{10}$ and $g(x) = x^3 - 7x^2 + 3$

By the Chain Rule: $F'(x) = \underbrace{10(x^3 - 7x^2 + 3)^9}_{f'(g(x))} \cdot \underbrace{(3x^2 - 7x)}_{g'(x)}$

Using Leibniz notation:

Let $y = f(u) = u^{10}$ and $u = g(x) = x^3 - 7x^2 + 3$

$$\frac{dy}{du} = 10u^9 \quad \text{and} \quad \frac{du}{dx} = 3x^2 - 7x$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = 10u^9(3x^2 - 7x) = 10(x^3 - 7x^2 + 3)(3x^2 - 7x)$$

Chain Rule in Use

(1) In applying the Chain Rule, we start at the outside and work to the inside.

(2) Differentiate the outer function f [at the inner function $g(x)$] and then multiply by the derivative of the inner function.

$$\frac{d}{dx}\underbrace{f}_{\substack{\text{outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{f'}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

(3) The Chain Rule: when we take the derivative of a composite function, start at the outside and work toward the inside, taking derivatives along the way.

After each derivative, we *peel* away that function, and take the derivative of the next innermost function.

Example 2 The Chain Rule and A Trigonometric Function

Differentiate (a) $y = \cos(x^2)$ and (b) $y = \cos^2 x$.

Solution

(a) $y = \cos(x^2)$: outer function: cosine; inner function: x^2

$$\frac{dy}{dx} = \frac{d}{dx} \underbrace{\cos}_{\text{outer function}} \underbrace{(x^2)}_{\text{evaluated at inner function}} = \underbrace{-\sin}_{\text{derivative of outer function}} \underbrace{(x^2)}_{\text{evaluated at inner function}} \cdot \underbrace{2x}_{\text{derivative of inner function}}$$

$$= -2x\sin(x^2)$$

Example 2 (Continued)

Solution

(b) $y = \cos^2 x = (\cos x)^2$:

outer function: squaring function; inner function: cosine function.

$$\frac{dy}{dx} = \frac{d}{dx}(\underbrace{\cos x}_{\text{inner function}})^2 = \underbrace{2 \quad \cdot}_{\text{derivative of outer function}} \underbrace{(\cos x)}_{\text{evaluated at inner function}} \cdot \underbrace{-\sin x}_{\text{derivative of inner function}}$$

$$= -2\sin x \cos x = -\sin 2x$$

The Power Rule and the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Or,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example 3

Let $f(x) = \dfrac{1}{\sqrt[5]{x^3 + x^2 + 4}}$. Find $f'(x)$.

Solution

Rewrite f: $f(x) = (x^3 + x^2 + 4)^{-1/5}$

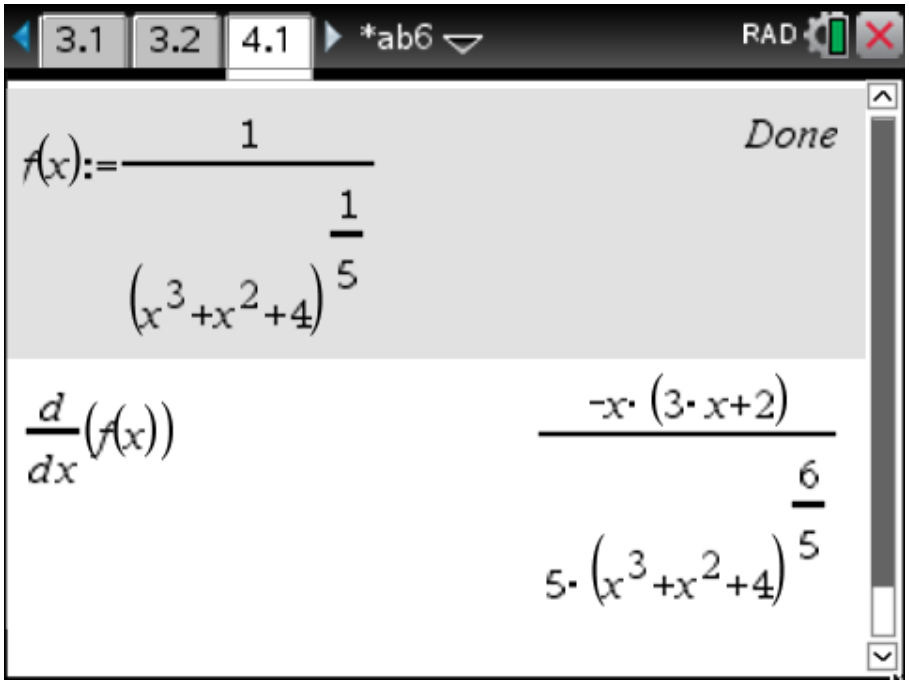
$$f'(x) = -\frac{1}{5}(x^3 + x^2 + 4)^{-6/5}\frac{d}{dx}(x^3 + x^2 + 4)$$

$$= -\frac{1}{5}(x^3 + x^2 + 4)^{-6/5}(3x^2 + 2x)$$

$$= \frac{-(3x^2 + 2x)}{5\sqrt[5]{(x^3 + x^2 + 4)^6}} = \frac{-(3x^2 + 2x)}{5(x^3 + x^2 + 4)\sqrt[5]{x^3 + x^2 + 4}}$$

Example 3 (Continued)

Technology Solution



Example 4 Product Rule and Chain Rule

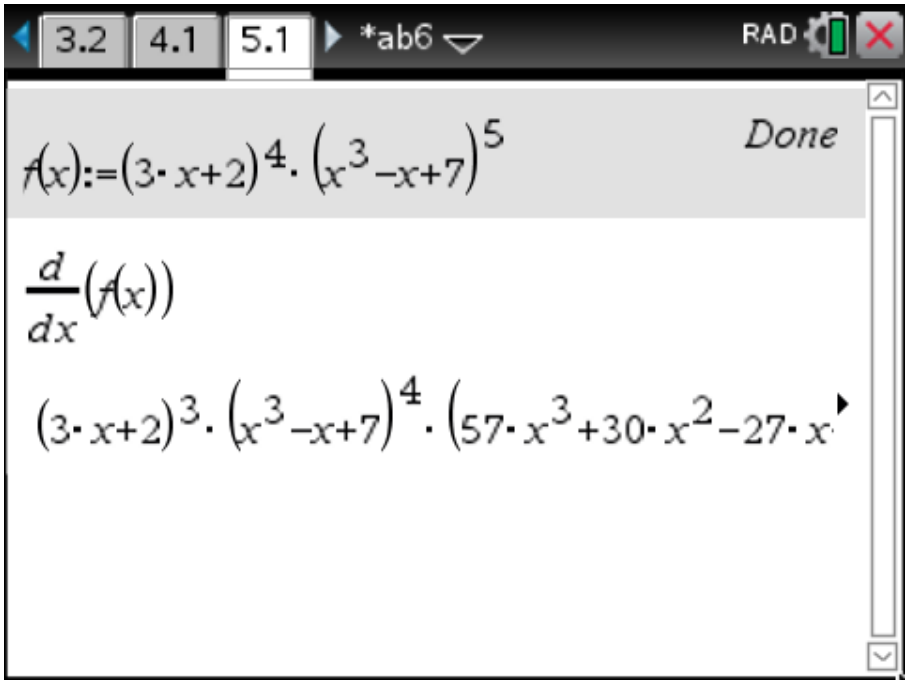
Find the derivative of $f(x) = (3x+2)^4(x^3 - x + 7)^5$.

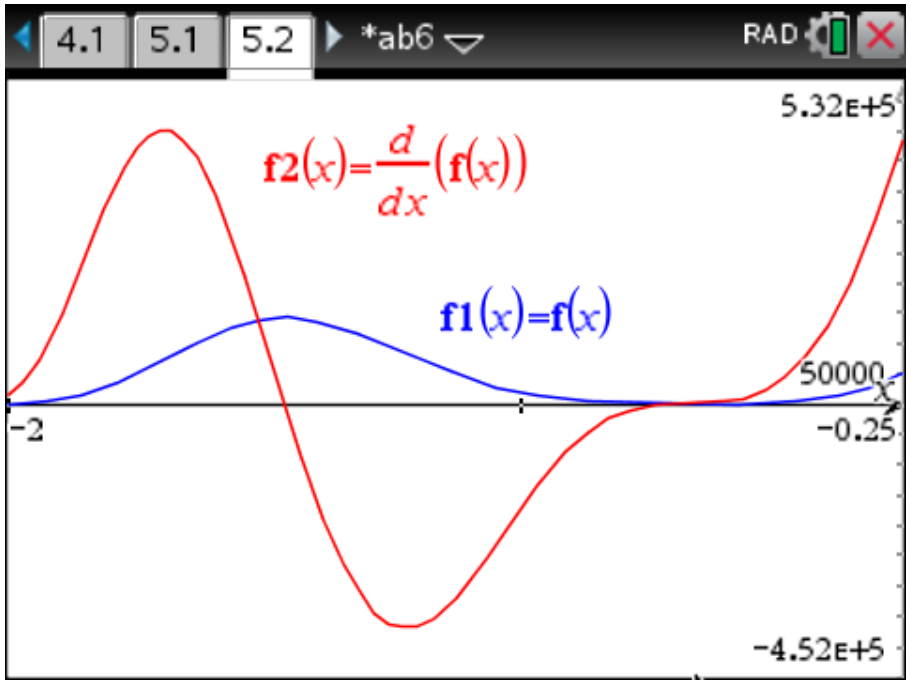
Solution

$$
\begin{aligned}
f'(x) &= (3x+2)^4 \frac{d}{dx}(x^3 - x + 7)^5 + \frac{d}{dx}(3x+2)^4(x^3 - x + 7)^5 \\
&= (3x+2)^4 \cdot 5(x^3 - x + 7)^4(3x^2 - 1) + 4(3x+2)^3(3) \cdot (x^3 - x + 7)^5 \\
&= (3x+2)^3(x^3 - x + 7)^4[(3x+2) \cdot 5(3x^2 - 1) + 12(x^3 - x + 7)] \\
&= (3x+2)^3(x^3 - x + 7)^4[57x^3 + 30x^2 - 27x + 74]
\end{aligned}
$$

Example 4 (Continued)

Technology Solution





Example 5 Exponential Functions and the Chain Rule

Find the derivative of $f(x) = e^{3x^2-5x+2}$.

Solution

The outer function is the exponential function, e^x.

The inner function is the polynomial function, $3x^2 - 5x + 2$

$$f'(x) = e^{3x^2-5x+2} \frac{d}{dx}(3x^2 - 5x + 2)$$

$$= (6x - 5)e^{3x^2-5x+2}$$

Example 6 Derivative, Move Inside, Repeat

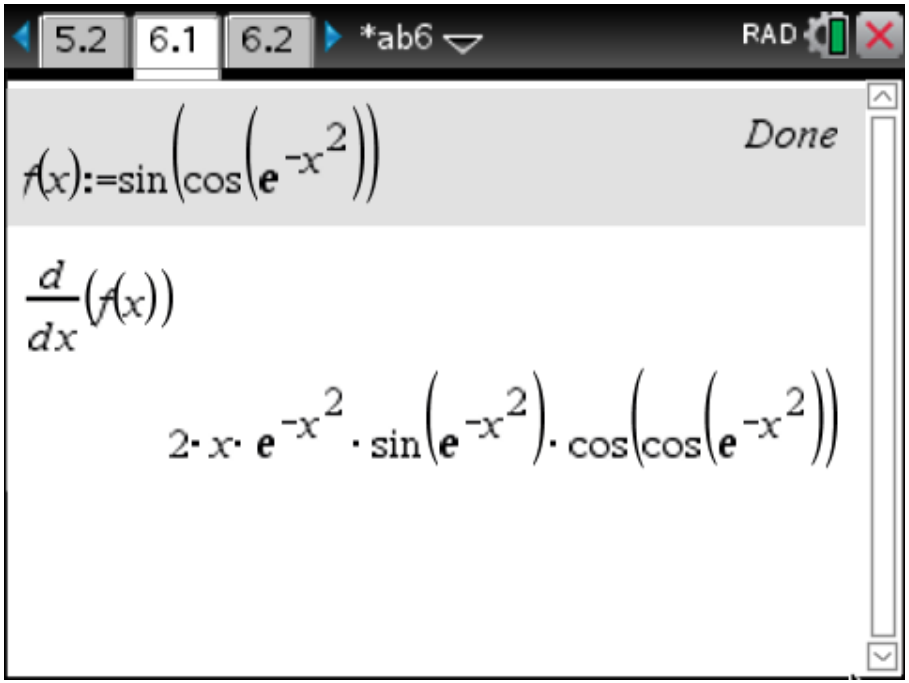
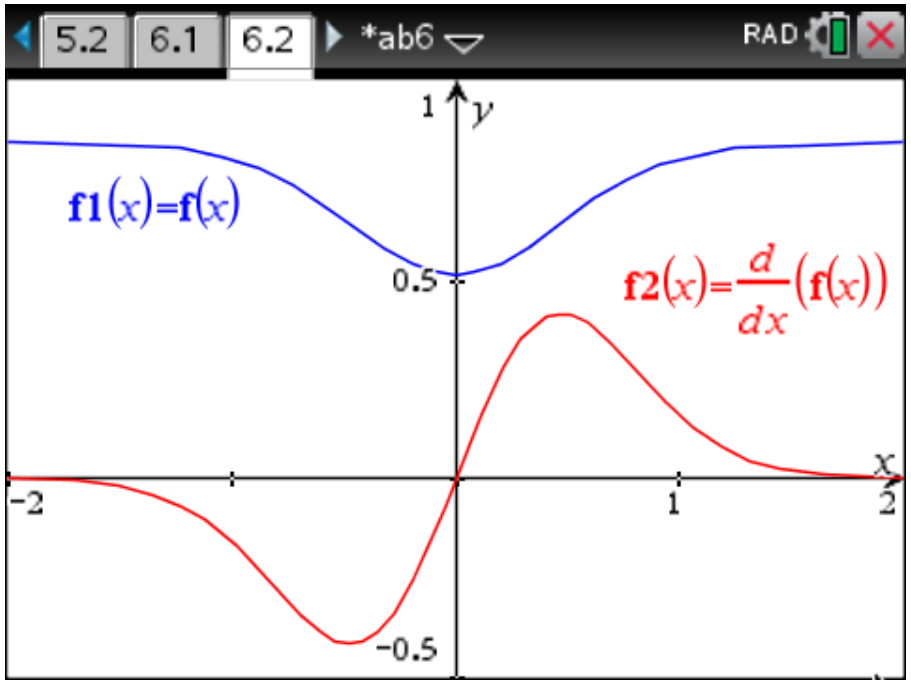
Find the derivative of $f(x) = \sin(\cos(e^{-x^2}))$.

Solution

THe outer most function is the sine function.

Working inside: there is a cosine function, an exponential function, and a power function.

$$
\begin{aligned}
f'(x) &= \cos(\cos(e^{-x^2})) \cdot \frac{d}{dx}(\cos(e^{-x^2})) \\
&= \cos(\cos(e^{-x^2})) \cdot (-\sin(e^{-x^2})) \cdot \frac{d}{dx}(e^{-x^2}) \\
&= \cos(\cos(e^{-x^2})) \cdot (-\sin(e^{-x^2})) \cdot e^{-x^2} \cdot \frac{d}{dx}(x^2) \\
&= \cos(\cos(e^{-x^2})) \cdot (-\sin(e^{-x^2})) \cdot e^{-x^2} \cdot (-2x) \\
&= 2xe^{-x^2} \cos(\cos(e^{-x^2})) \cdot (\sin(e^{-x^2}))
\end{aligned}
$$

Example 6 (Continued)

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