Monday Night Calculus, October 18, 2021

1. $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$ (Sarah Strick)

Definition: Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists as a finite number.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

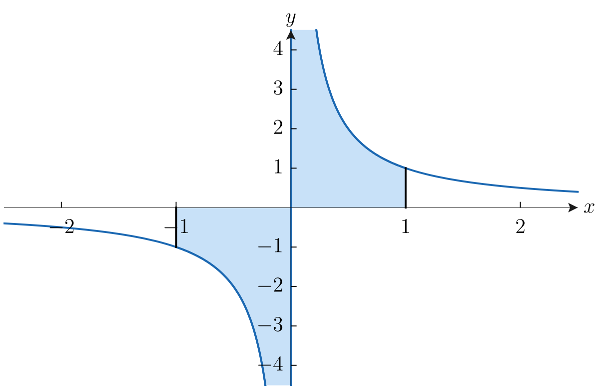
if this limit exists as a finite number.

The improper integral $\displaystyle\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$ and both $\displaystyle\int_a^c f(x)\,dx$ and $\displaystyle\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$



$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx \qquad \text{Improper integral definition}$$

$$= \lim_{t\to 0^+} \Big[\ln|x|\Big]_t^1 \qquad \text{Antiderivative}$$

$$= \lim_{t\to 0^+} [\ln 1 - \ln t] \qquad \text{FTC}$$

$$= \lim_{t\to 0^+} (-\ln t) = \infty \qquad \text{Evaluate limit}$$

$$\int_0^1 \frac{1}{x}\,dx \text{ diverges} \;\Rightarrow\; \int_{-1}^1 \frac{1}{x}\,dx \text{ diverges.}$$

2. Intervals on which a function is increasing or decreasing, concave up, or concave down: endpoints. (Dorothy Buddy Rich)

Definition

A function f is **increasing** on an interval I if for any values x_1 and x_2 in I, with $x_1 < x_2$, then $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval I if for any values x_1 and x_2 in I, with $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Note: This definition is in terms of an interval, not a value.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example Increasing/Decreasing

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$

Candidates for extrema:

$f'(x) = 0: \quad x = -1,\ 0,\ 2$

$f'(x)$ DNE: none

$-$	0	$+$	0	$-$	0	$+$	$f'(x)$
	-1		0		2		x
$\downarrow$		$\uparrow$		$\downarrow$		$\uparrow$	$f(x)$

f increasing: $[-1, 0],\ [2, \infty)$

f decreasing: $(-\infty, -1],\ [0, 2]$

Exam Scoring

Endpoints do not matter, unless:

The function is undefined.

Closed at infinity: $(10, \infty]$

Definition

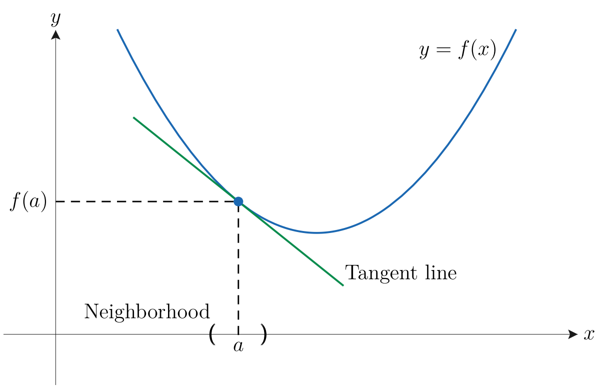
Let f be a differentiable function.

f is **concave up** at a if the graph of f is above the tangent line to f at a for all x in a neighborhood of a (but not equal to a).

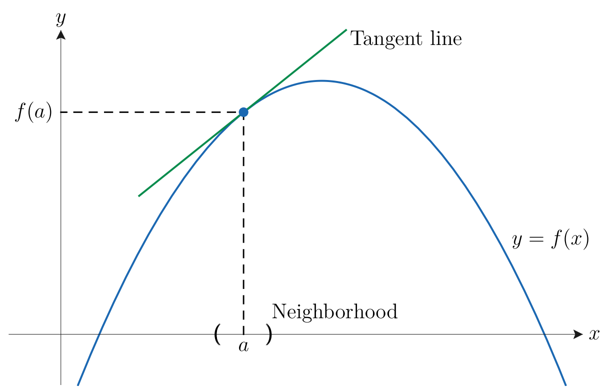
f is **concave down** at a if the graph of f is below the tangent line to f at a for all x in a neighborhood of a (but not equal to a).

Note: This definition is in terms of a specific value, not an interval.

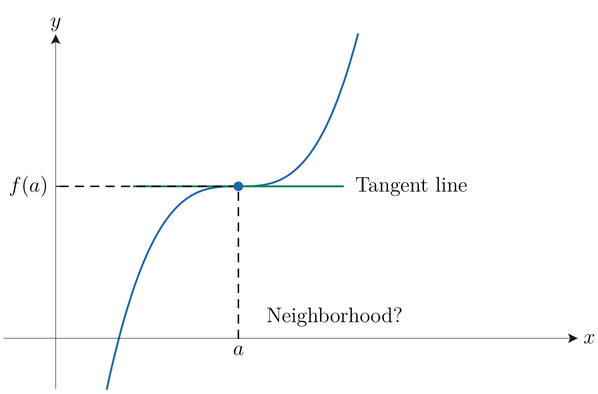
Concave Up



Concave Down



No Concavity



Example Concavity and Points of Inflection

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity and points of inflection.

Solution

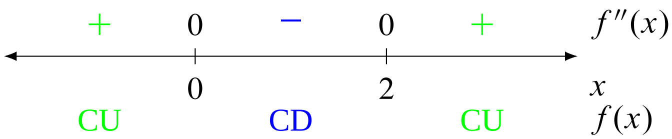
$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

Candidates for points of inflection:

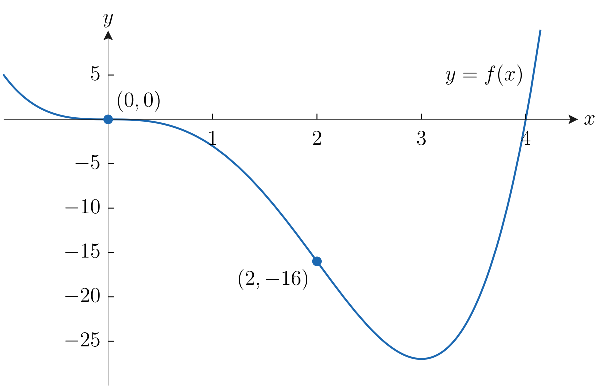
$f''(x) = 0$: $x = 0,\ 2$

$f''(x)$ DNE: none



Concave up: $(-\infty, 0)$, $(2, \infty)$

Concave down: $(0, 2)$



Inflection Points:

$(0, f(0)) = (0, 0)$; $(2, f(2)) = (2, -16)$

Scoring Conclusion

1. Inclusion or exclusion of endpoints do not matter unless there is a contradiction.

2. A sign chart is not sufficient justification.

3. Written justification (confirmation of a sign chart) is necessary in order to receive credit.

Definition: Inflection Point

A point P on the graph of f is called an **inflection point** (IP) if f is continuous there and the graph changes from concave up to concave down or from concave down to concave up at P.

A Closer Look

1. If $f''(a)$ exists and $f''(a) \neq 0$: concavity is known, graph cannot change concavity at $(a, f(a))$.

   $f''(x)$ can change sign only when $f''(x) = 0$ or $f''(x)$ DNE.

2. Concavity Test: IP only where second derivative changes sign.

   Use a sign chart for the second derivative.

Procedure for Determining Inflection Points

1. Find the IP candidates:

   Those x in the domain of f such that $f''(x) = 0$ or $f''(x)$ DNE.

2. Screen the IP candidates:

   Check for a change in sign of f'' at each candidate.

   If a change in sign occurs, then $(x, f(x))$ is a point of inflection.

   If no change in sign, then $(x, f(x))$ is not a point of inflection.

3. The differentiable functions p and q are defined for all real numbers x. Values of p, p', q, and q' for various values of x are given in the table.

x	$p(x)$	$p'(x)$	$q(x)$	$q'(x)$
4	10	8	4	2
5	4	9	16	7

(a) If $f(x) = p(\sqrt{q(x)})$, find $f'(5)$.

(b) If $h(x) = \dfrac{q(x)}{x}$, find $h'(4)$.

Solution

(a) $f(x) = p(\sqrt{q(x)}) \quad \Rightarrow \quad f'(x) = p'(\sqrt{q(x)}) \cdot \dfrac{1}{2} q(x)^{-1/2} \cdot q'(x)$

$$f'(5) = p'(\sqrt{q(5)}) \cdot \frac{1}{2} q(5)^{-1/2} \cdot q'(5)$$

$$= p'(\sqrt{16}) \cdot \frac{1}{2\sqrt{16}} \cdot 7$$

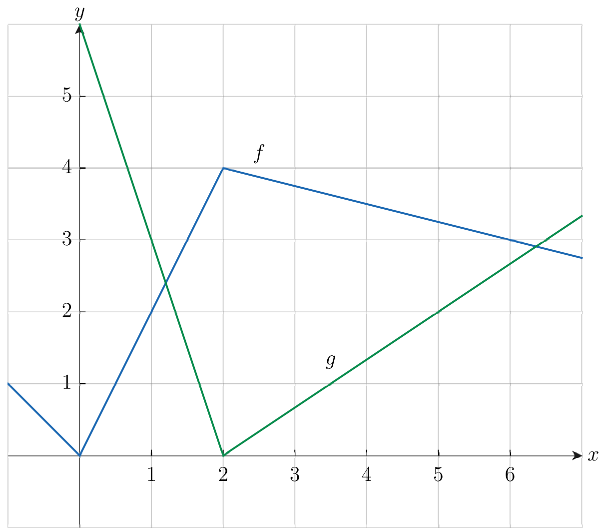
$$= p'(4) \cdot \frac{1}{8} \cdot 7 = 8 \cdot \frac{1}{8} \cdot 7 = 7$$

(b) $h(x) = \dfrac{q(x)}{x} \quad \Rightarrow \quad h'(x) = \dfrac{xq'(x) - q(x) \cdot 1}{x^2}$

$$h'(4) = \frac{4 \cdot q'(4) - q(4)}{4^2}$$

$$= \frac{4 \cdot 2 - 4}{16} = \frac{4}{16} = \frac{1}{4}$$

4. The graphs of the functions f and g are shown in the figure.



Let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$.
Find each derivative if it exists.

(a) $u'(1)$ **(b)** $v'(1)$ **(c)** $w'(1)$

Solution

(a) $u'(x) = f'(g(x)) \cdot g'(x)$

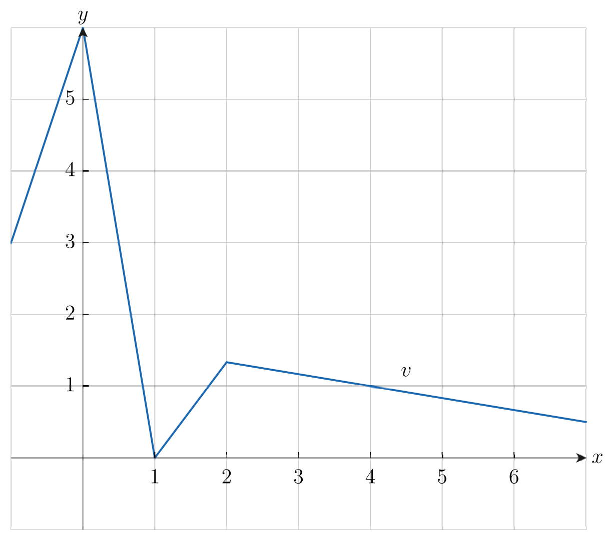
$$u'(1) = f'(g(1)) \cdot g'(1) = f'(3) \cdot (-3)$$

$$= -\frac{1}{4} \cdot (-3) = \frac{3}{4}$$

(b) $v'(x) = g'(f(x)) \cdot f'(x)$

$$v'(1) = g'(f(1)) \cdot f'(1) = g'(2) \cdot 2$$

$g'(2)$ does not exist.



$v'(1)$ does not exist. Can you show this analytically?

(c) $w'(x) = g'(g(x)) \cdot g'(x)$

$$w'(1) = g'(g(1)) \cdot g'(1) = g'(3) \cdot (-3)$$

$$= \frac{2}{3} \cdot (-3) = -2$$