How Many Solutions to the System?

Math Objectives

- Students will understand that a system of two linear equations in two variables can have one solution, no solution, or infinitely many solutions.
- Students will understand the connection between the slopes of the lines and the number of solutions to a system of linear equations in two variables.
- Students will look for regularity in repeated reasoning. (CCSS Mathematical Practice)

Vocabulary

- system of two linear equations
- solution

About the Lesson

- This lesson involves graphing systems of linear equations. The emphasis is on helping students understand the difference between systems that have one, infinitely many, or no solutions.
- As a result, students will:
  - Manipulate a movable line in the coordinate plane in relation to a fixed line to satisfy certain conditions.
  - Observe the slope and *y*-intercept changing as they manipulate the line.
  - Discover what must be true for a system of equations to have one, infinitely many, or no solutions.

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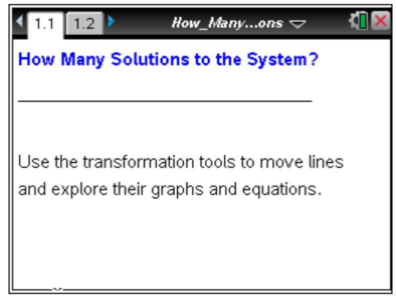
- Use **Quick Poll** to check student understanding.
- Use **Screen Capture** to examine patterns that emerge.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire™ document
- Open a document
- Move between pages
- Rotate and translate a movable line

Tech Tips:

- Make sure the font size on your TI-Nspire™ handheld is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Materials:
Student Activity
How_Many_Solutions_Student.pdf
How_Many_Solutions_Student.doc

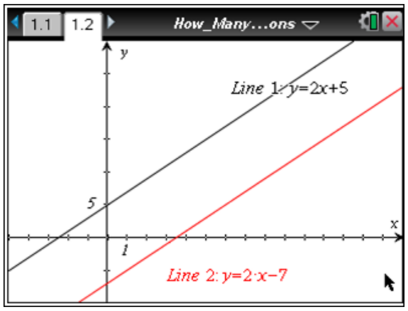
TI-Nspire document
How_Many_Solutions.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

Tech Tip: If students experience difficulty dragging the line, check to make sure that they have moved the arrow until it becomes a rotation symbol (↻) or a translation symbol (✥). When finished moving the line, press esc to release the line.

Move to page 1.2.



1. a. As you rotate (↻) Line 2, describe the changes you observe in its graph and its equation.

   Answer: As the line is rotated, the coefficient of *x* in the equation for Line 2 changes, and the steepness of the graph changes.

   b. As you translate (✥) Line 2, describe the changes you observe in its graph and its equation.

   Answer: As the line is translated, the constant value in the equation for Line 2 changes, and the *y*-intercept of the graph changes.

Teacher Tip: It might be important to briefly review the meaning of slope and how it relates to the steepness of the line. Note that if students translate and end up with a labeled slope of 2.0, it is not the same, in this case, as a slope of exactly 2. Here the 2.0 represents a rounded value and could actually be anywhere between 1.95 and 2.04.

2. Move Line 2 so that it has exactly one point in common with Line 1. If you make the slope of Line 2 the same as the slope of Line 1, can the lines still have only one point in common? Explain.

   Answer: If two lines have the same slope, either they will lie on top of each other and have all of their points in common, or they will be parallel and have no points in common. For the lines to have exactly one point in common, Line 2 must have a slope other than 2 (the slope of Line 1).

TI-Nspire Navigator Opportunity: ***Screen Capture***
See Note 1 at the end of this lesson.

> **Teacher Tip:** Students may have different equations, but they should notice that as long as the slope of Line 2 is not 2, their lines will have exactly one intersection point. Watch for students who think that the only way two lines can intersect is for one to have a negative slope and the other a positive slope. Have these students move Line 2 so that the two lines intersect and both slopes are positive. You can also ask the students to make observations or comments about the *y*-intercepts. Some might think that the *y*-intercepts must be different for the lines to intersect. Ask these students to move Line 2 so that it intersects Line 1 at its *y*-intercept. Guide the discussion toward the idea that having different slopes is the only condition that is necessary for two lines to intersect. Also, the lines might look like they have more than one point in common due to the thickness of the lines and the scale. Remind students that lines actually have no width.
>
> It might be necessary to talk about how lines have a constant rate of change and how if two lines have different rates of change, they will have exactly one point in common.

3. Move Line 2 so the lines do not have any points in common. How can you be certain these lines never intersect?

   Sample answers: The slopes of Line 1 and Line 2 are exactly the same, and the *y*-intercepts are different, so the two lines are parallel. We know that parallel lines never intersect.

> **Teacher Tip:** It is crucial for students to explain how they know the lines are parallel using slope and *y*-intercept. Students might think some lines are parallel because they “look” parallel, or because they do not intersect within the current window settings. Ask questions that lead students to describe the graph outside the given window. If the two lines do not have exactly the same slope, the lines will eventually intersect, but if the slopes are the same (and the *y*-intercepts are different), they will never intersect.

Suppose two lines have the same slope: $y = 2x + a$ and $y = 2x + b$. If a and b are different, the numerical values for y will always be different for the same value of x. Thus, for any value of x there is no way to have a common point (x, y) that is on both lines.

4. The point of intersection of two lines is a solution to a system of equations. How is the graph of a linear system with no solution different from the graph of a linear system with only one solution?

   Answer: If a linear system has no solution, the two lines are parallel. If it has one solution, the lines will intersect at exactly one point because they have different slopes.

   Teacher Tip: To help students understand that an intersection point is a solution to a system, remind them that every point on a line is a solution to its corresponding equation because when the ordered pair is substituted into the equation of the line, it makes the equation true. Since a point of intersection is on both lines, it must be a solution to both equations.

5. Joel says a system of linear equations will always have exactly one solution whenever the slopes of the two lines are different. Is Joel correct? Why or why not?

   Answer: Joel is correct. If the slopes of the two lines are different, the lines are not parallel and are not the same line. Therefore, they must intersect at one and only one point.

   Teacher Tip: Even though this question might seem a repeat of question 2, it is included here for reinforcement, since the message is somewhat more personalized when put into this kind of context.

   Note that if $y = mx + b$ and $y = cx + d$, then $mx + b = cx + d$. To solve for x: $(m - c)x = d - b$. If $m \neq c$, you will have one value for x: $(d - b)/(m - c)$. That value will give you one corresponding value for y. Thus, the two lines intersect at exactly one point.

6. a. Move Line 2 so that there is more than one point of intersection with Line 1. What do you observe about the two lines?

   Answer: Two unique lines cannot possibly have more than one intersection point because two distinct points determine a unique line, so the lines would have to coincide. Therefore, they must have the same slope and the same y-intercept.

Teacher Tip: Some students might have the misconception that if two lines have slopes that are very close to each other, there can be a one- or two-inch "interval" of overlap that contains many intersection points. Remind them that lines are made up of points that are infinitesimally small, and even though it might look like two lines touch each other at several points over an interval, the settings on the calculator can prevent us from seeing them as they really are. You might want to illustrate with a class demonstration where you zoom in closer and closer to such intersecting lines.

Students might also suggest "bending" or "curving" one of the lines to make the system have two solutions. This could be a nice entry into a discussion of nonlinear systems.

b. How many solutions are there to the system represented by two lines that have more than one point of intersection? Explain your reasoning.

Answer: The system has infinitely many solutions because the two equations representing the lines are equivalent. The lines coincide. Thus, all of the ordered pairs that are solutions for one equation are solutions for the other. A linear equation in two variables has an infinite number of ordered pairs in its solution set, so there are infinitely many ordered pairs (points) in common between the two coinciding lines.

Teacher Tip: Some students might have the misconception that infinitely many solutions means any ordered pair is a solution to the system. When two lines coincide, every point on one line will be on the other; therefore, they have an infinite number of points in common. A linear system with infinitely many solutions means every coordinate point that is on the line, determined by the two equivalent equations, is a solution to the system.

TI-Nspire™ Navigator™ Opportunity: *Quick Poll (Multiple Choice or Open Response)* See Note 2 at the end of this lesson.

7. Given a system in which one of the equations is $y = -7x + 4$, create a second equation such that the resulting system has:

   a. Exactly one solution

Sample answers: The slope of the line must not be –7. Examples: $y = 2x - 3$, $y = 5x + 12$.

b. No solution

Sample answers: The slope of the line must be –7, and the *y*-intercept must not be 4. Examples: $y = -7x - 3$, $y = -7x + 8.2$.

c. Infinitely many solutions

Sample answers: Answers must be equivalent to $y = -7x + 4$. Examples: $2y = -14x + 8$, $-y = 7x - 4$.

8. What could you say to convince another student that your answers to question 7 are correct?

Answer: Students might graph an example for each question. Students should justify their answers in terms of slopes and *y*-intercepts.

> **Teacher Tip:** This question can promote good discussion between partners. Both partners might be correct even though they might have different answers. Some students might rearrange the equation ($7x + y = 4$), or they might multiply the existing equation by a common multiplier. The class as a whole might discuss these techniques, remembering that equivalent equations are those that have identical solution sets. Both suggestions above would be acceptable.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students are able to understand:

- The graph of a linear system having exactly one solution contains lines that intersect at only one point and have different slopes.
- The graph of a linear system having no solution contains parallel lines with the same slope and different *y*-intercepts.
- The graph of a linear system having infinitely many solutions contains coinciding lines that have the same slope and the same *y*-intercept.

Assessment

- Questions 7 and 8 are targeted specifically at assessing whether students understand the main concepts of the lesson.
- You can also use a graphic organizer similar to the one shown below for the students to complete by checking the box for each possible combination.

Linear system solutions	Same slope	Different slope	Same *y*-intercept	Different *y*-intercepts	Intersecting lines	Parallel lines	Coinciding lines
No solution							
One solution							
Infinitely many solutions							

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Note 1

Question 2, *Screen Capture*: Use **Screen Capture** to examine pairs of lines generated by students. Compare lines with and without rotations.

Note 2

Question 6, *Quick Poll (Multiple Choice or Open Response)*: Send students a **Quick Poll** asking for the total number of points of intersection of the two lines. For more advanced students, use an Open Response **Quick Poll**. For more remedial students, send a Multiple Choice **Quick Poll** with possible responses **0**, **1**, **2**, and **Infinitely many**.