

TI in Focus: AP[®] Calculus

2017 AP[®] Calculus Exam: AB-4/BC-4
Separable Equations

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Outline

- (1) Separable equation definition
- (2) Solution technique
- (3) Examples
- (4) Interval of definition; maximum interval of convergence

Background

- (1) First-order differential equations:
 - (a) Geometrically: slope fields
 - (b) Numerically: Euler's method
- (2) Consider a symbolic approach: find an explicit formula for a solution to a differential equation.
- (3) Note: cannot always find an explicit solution.
- (4) In certain situations, it may be possible to solve explicitly for the dependent variable.

Definition

A **separable equation** is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y . That is, the differential equation can be written in the form

$$\frac{dy}{dx} = g(x) f(y)$$

A Closer Look

- (1) Separable: the expression on the right side can be *separated* into a function of x and a function of y .
- (2) If $f(y) \neq 0$, then the differential equation can be written as

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = \frac{1}{f(y)}$

Solve the Differential Equation

- (1) Write the differential equation as

$$h(y) dy = g(x) dx$$

- (2) Integrate both sides of the equation:

$$\int h(y) dy = \int g(x) dx$$

This equation defines y implicitly as a function of x .

We may be able to solve explicitly for y in terms of x .

- (3) The Chain Rule is used to justify this procedure.

Example 1 Principle of Separation

- (a) Solve the differential equation $xy^2 y' = 3x^3 + 2x^2$
- (b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Solution

$$xy^2 \frac{dy}{dx} = 3x^3 + 2x^2$$

Write $y' = \frac{dy}{dx}$

$$y^2 dy = \frac{3x^3 + 2x^2}{x} dx = (3x^2 + 2x) dx$$

Separate the variables

$$\int y^2 dy = \int (3x^2 + 2x) dx$$

Integrate both sides

$$\frac{y^3}{3} = x^3 + x^2 + C$$

Integrate as indicated by the differentials

$$y = \sqrt[3]{3x^3 + 3x^2 + C}$$

Solve for y

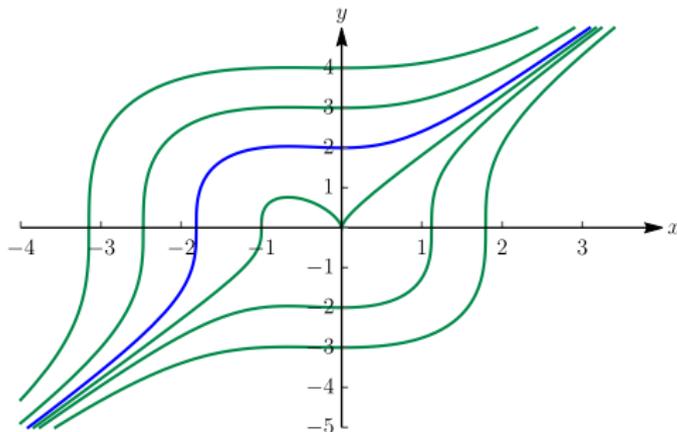
Solution

$$y(0) = \sqrt[3]{3 \cdot 0^3 + 3 \cdot 0^2 + C} = \sqrt[3]{C} = 2 \implies C = 8$$

$$y = \sqrt[3]{3x^3 + 3x^2 + 8}$$

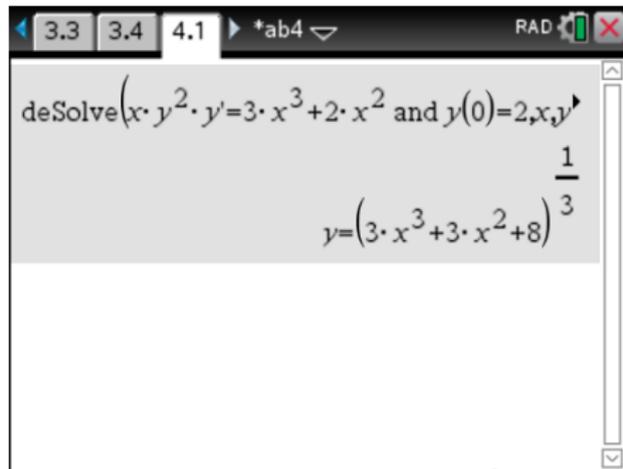
Note

- (1) It is often easier to determine the constant of integration, and therefore, a particular solution, immediately after integrating.
- (2) Graphs of several members of the family of solutions of the differential equation.

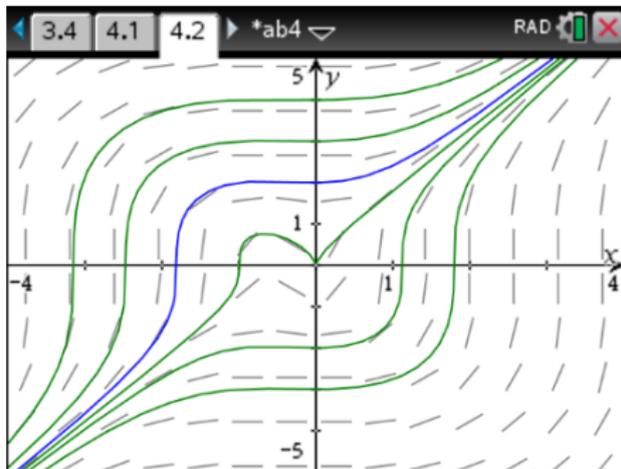


Solution

A solution to the differential equation that satisfies the initial condition $y(0) = 2$.



The slope field and graphs of members of the family of solutions of the differential equation.



Example 2 Initial-Value Problem

Consider the differential equation $y' = yx \cos x$. Find the particular solution to the differential equation that satisfies the initial condition.

(a) $y(0) = 2e$.

(b) $y(0) = -e$.

Solution

$$\frac{dy}{y} = x \cos x \, dx$$

Separate the variables

$$\int \frac{dy}{y} = \int x \cos x \, dx$$

Integrate both sides

$$\ln |y| = \cos x + x \sin x + C$$

Basic antiderivative formula; parts

Solution

$$(a) \ln |2e| = \cos 0 + 0 \cdot \sin 0 + C = 1 + C$$

Use initial value

$$\ln 2e = \ln 2 + \ln e = 1 + C \implies C = \ln 2$$

Solve for C

$$\ln y = \cos x + x \sin x + \ln 2$$

Because $y(0) = 2e$, $y > 0$ and
 $|y| = y$

$$y = 2e^{\cos x + x \sin x}$$

Solve explicitly for y

$$(b) \ln |-e| = \cos 0 + 0 \cdot \sin 0 + C = 1 + C$$

Use initial value

$$\ln e = 1 + C \implies C = 0$$

Solve for C

$$\ln(-y) = \cos x + x \sin x$$

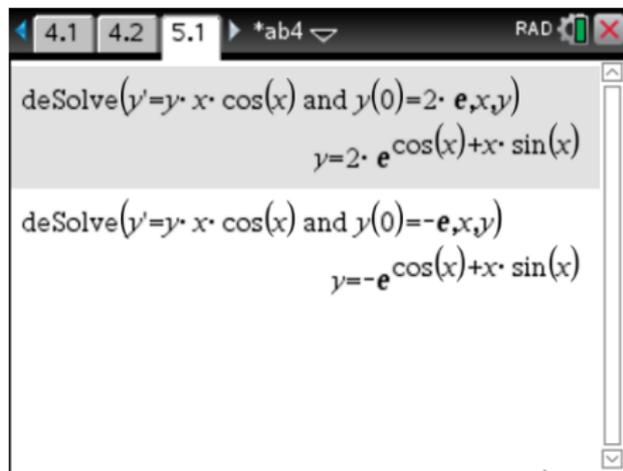
Because $y(0) = -e$, $y < 0$ and $|y| = -y$

$$y = -e^{\cos x + x \sin x}$$

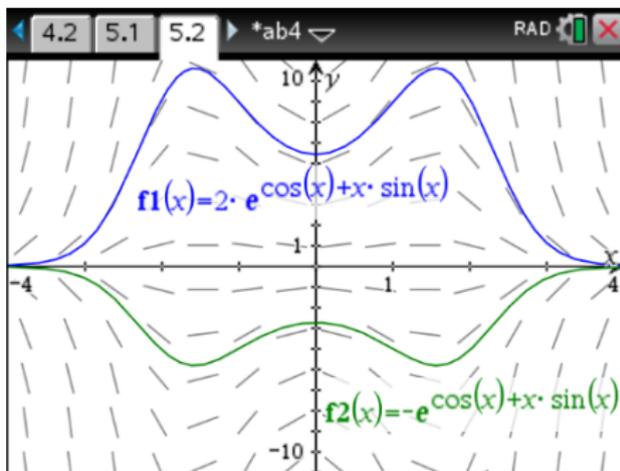
Solve explicitly for y

Solution

Solution to the differential equation using the initial condition.



Graphs of the particular solutions.



Question: Describe the behavior of the graph of the particular solution as x increases (decreases) without bound, that is, as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

Other Issues

- (1) What if the solution involved a natural logarithm of a function of x ?
- (2) After separating the variables, one or both sides may not have a closed form antiderivative.
- (3) Applicable to particle motion problems: write the acceleration (velocity) function as a differential equation, separate the variables, and integrate both sides.
- (4) AP[®] Calculus Exam: lots of applied, contextual questions involving separable equations.

Example 3 Best Known Name in Sweet Corn

Silver Queen corn is popular due to its sweet flavor, appealing white kernels, and wide adaptability. For a farmer in New Jersey, at time $t = 0$ when a seedling is planted, the height of the stalk is 0.5 feet. If $h(t)$ is the height of the stalk, in feet, at time t days after it is planted, then the rate of growth is given by

$$\frac{dh}{dt} = \frac{1}{25}(8 - h)$$

where $h < 8$.

- Is the corn stalk growing faster when it is 2 feet or 4 feet? Explain your reasoning.
- Find $\frac{d^2h}{dt^2}$ in terms of h . Use this expression to explain why the graph of the solution curve is concave down.
- Find an equation of the tangent line to the graph of h at the point where $t = 0$. Use the tangent line to approximate the height of the corn stalk at $t = 10$. Is the estimate an overestimate or an underestimate? Explain your reasoning.
- Find the particular solution to this differential equation with initial value $h(0) = 0.5$.
- If the corn is harvested when the stalk is 7.7 feet tall, how long does it take before the corn is ready to sell?

Solution

(a) Evaluate the first derivative at $h = 2$ and $h = 4$.

$$\left. \frac{dh}{dt} \right|_{h=2} = \frac{1}{25}(8 - 2) = \frac{6}{25}$$

$$\left. \frac{dh}{dt} \right|_{h=4} = \frac{1}{25}(8 - 4) = \frac{4}{25}$$

Since $\left. \frac{dh}{dt} \right|_{h=2} > \left. \frac{dh}{dt} \right|_{h=4}$

the corn stalk is growing faster when it is 2 feet tall.

Solution

$$(b) \frac{d^2h}{dt^2} = \frac{1}{25}(-1) \frac{dh}{dt} = -\frac{1}{25} \cdot \frac{1}{25}(8-h) = -\frac{1}{625}(8-h)$$

Since $\frac{d^2h}{dt^2} < 0$, the graph of the solution curve is concave down for $t > 0$.

$$(c) h(0) = \frac{1}{2}, \quad \left. \frac{dh}{dt} \right|_{h=\frac{1}{2}} = \frac{1}{25} \left(8 - \frac{1}{2} \right) = \frac{3}{10}$$

An equation of the tangent line: $y - \frac{1}{2} = \frac{3}{10}(t - 0) \implies y = \frac{3}{10}t + \frac{1}{2}$

$$\text{Estimate: } y(10) = \frac{3}{10}(10) + \frac{1}{2} = \frac{7}{2} = 3.5$$

Since the graph of h is concave down, the tangent line is above the graph of h , and therefore, this is an overestimate.

Solution

$$(d) \quad \frac{dh}{dt} = \frac{1}{25}(8 - h)$$

$$\int \frac{1}{8 - h} dh = \frac{1}{25} dt$$

Separate the variables

$$-\ln |8 - h| = \frac{1}{25}t + C$$

Integrate both sides

Since $0.5 \leq h < 8$, $|8 - h| = 8 - h$.

$$-\ln(8 - 0.5) = \frac{1}{25}(0) + C \implies -\ln 7.5 = C$$

$$h = 8 - 7.5e^{-t/25}$$

Solve for h

Solution

(e) Find the value of t for which $h(t) = 7.7$.

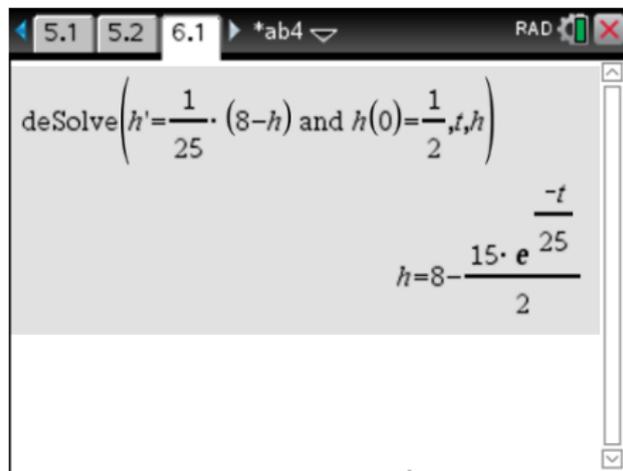
$$\frac{77}{10} = 8 - \frac{75}{10}e^{-t/25}$$

$$\frac{1}{25} = e^{-t/25}$$

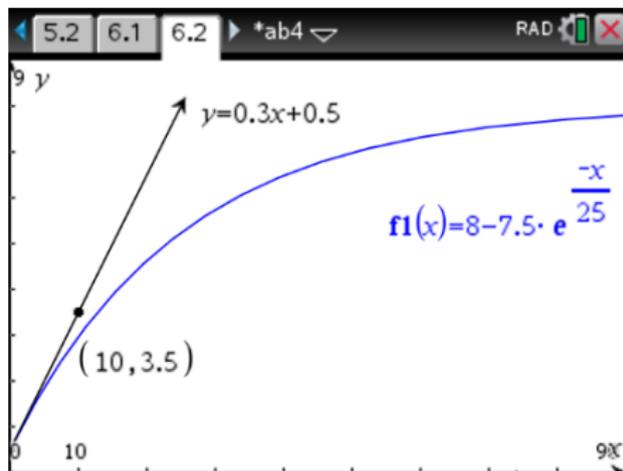
$$t = -25 \ln \left(\frac{1}{25} \right) = -25(-2) \ln 5 = 50 \ln 5 = 80.472 \text{ days}$$

Solution

Solution to the differential equation using the initial condition.

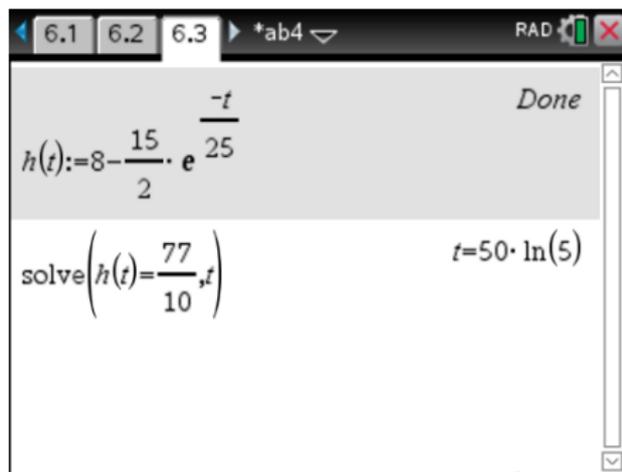


Graph of the particular solution and the tangent line.

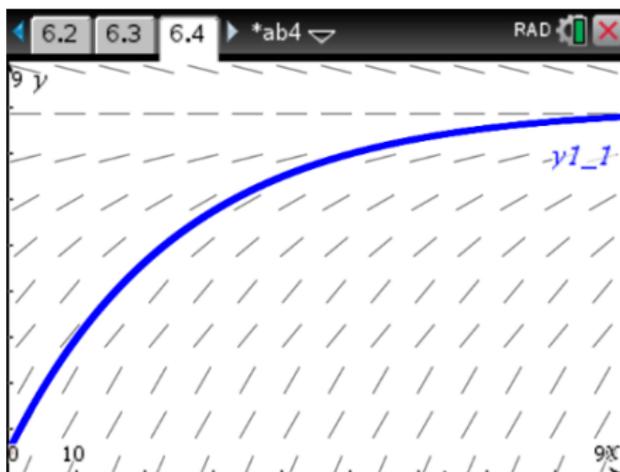


Solution

Harvest time.



The slope field.



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