

TI in Focus: AP[®] Calculus

2017 AP[®] Calculus Exam: AB-2

The Definite Integral as an Accumulation Function

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Outline

- (1) The Fundamental Theorem of Calculus
- (2) An Alternate Interpretation
- (3) Examples

Important AP Calculus Concepts

- The definite integral as an accumulation function.
- The Net Change Theorem.

The Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.

(1) If $g(x) = \int_a^x f(t) dt$, then $g'(x) = f(x)$.

(2) $\int_a^b f(x) dx = F(b) - F(a)$ where F is any antiderivative of f ,
that is, $F' = f$.

A Closer Look

- (1) The function g in the FTC depends only on x .
If x is a fixed number, the integral is a number.
If x varies, the integral defines a function.
- (2) If f is a positive function: $g(x)$ is the area under the graph of f from a to x .
The function g is the *area so far*, or *accumulation* function.
If f is continuous on $[a, b]$, positive and negative, g is the *net* area so far function.
- (3) The FTC provides a much simpler method for evaluating a definite integral.

If F is any antiderivative of f :
$$\int_a^b f(x) dx = F(b) - F(a)$$

Alternate Interpretation

- $F'(x)$: represents the rate of change of $y = F(x)$ with respect to x .
- $F(b) - F(a)$: the change in y as x changes from a to b .
- But: y could, for example, increase, then decrease, and then increase again.
- $F(b) - F(a)$: the *net* change in y .

Net Change Theorem

The definite integral of a rate of change, F' , is the net change in the original function F :

$$\int_a^b F'(x) dx = F(b) - F(a)$$

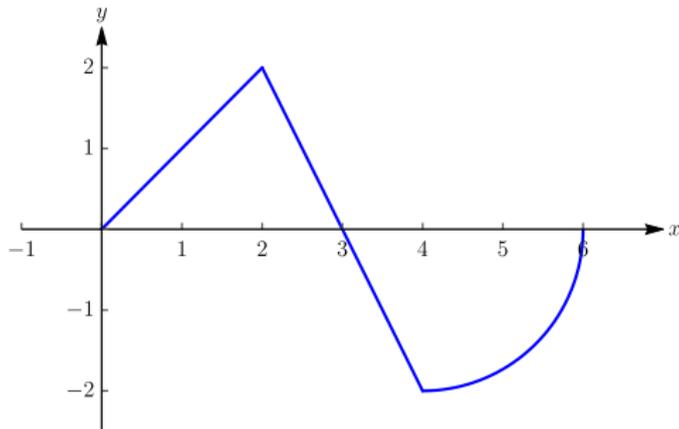
Still Another Interpretation

- $\int_a^b F'(x) dx$: an accumulation of the change in F over the interval $[a, b]$.
- Rearrange the terms in the Net Change Theorem:

$$\underbrace{F(b)}_{\text{End amount}} = \underbrace{F(a)}_{\text{Start amount}} + \underbrace{\int_a^b F'(x) dx}_{\text{Net change}}$$

Example 1 Net Area So Far

Suppose f is the function whose graph consists of straight line segments and a quarter circle, as shown in the figure. Define $g(x) = \int_0^x f(t) dt$.



- Find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(6)$.
- Sketch a rough graph of g .

Solution

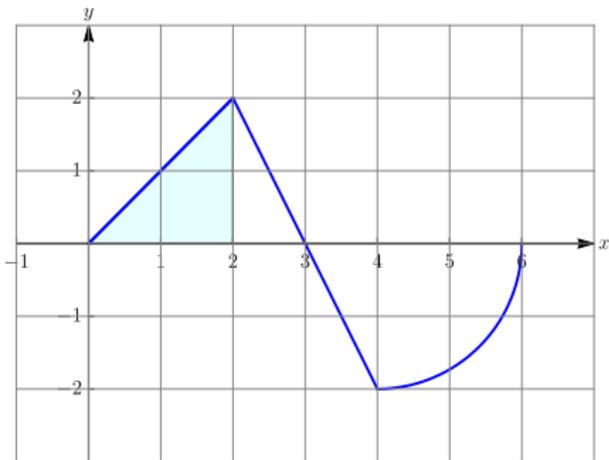
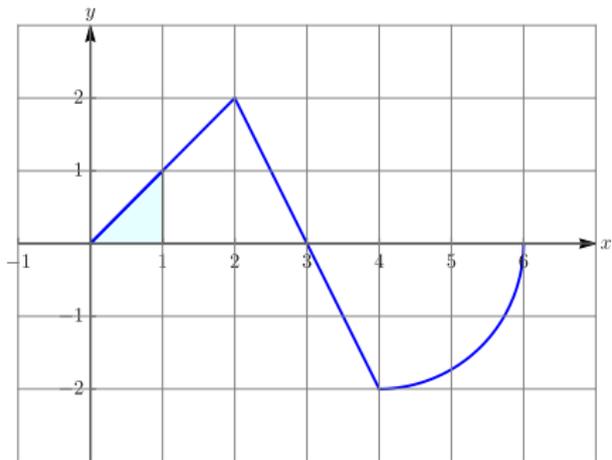
The function $g(x)$ can be interpreted as the net area of the region bounded by the graph of f , the x -axis, over the interval $[0, x]$.

$$g(0) = \int_0^0 f(t) dt = 0$$

$$g(1) = \int_0^1 f(t) dt = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

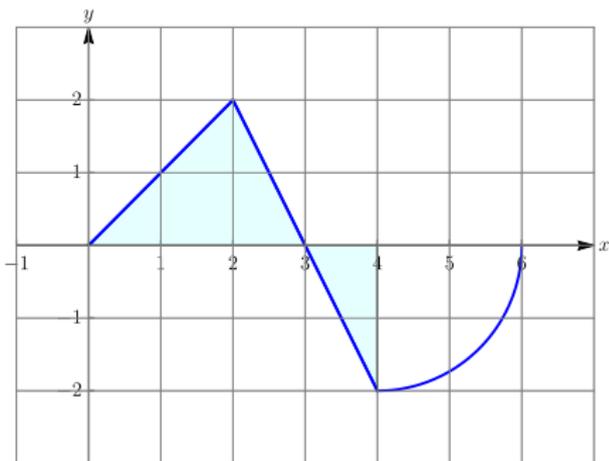
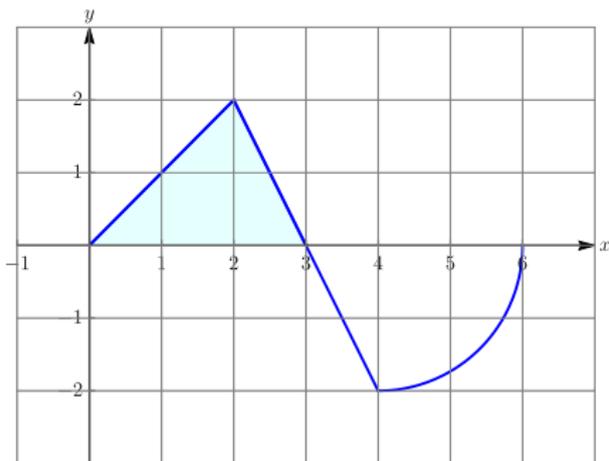
$$g(2) = \int_0^2 f(t) dt = \frac{1}{2} \cdot 2 \cdot 2 = 2$$

Geometric illustrations for $g(1)$ and $g(2)$.



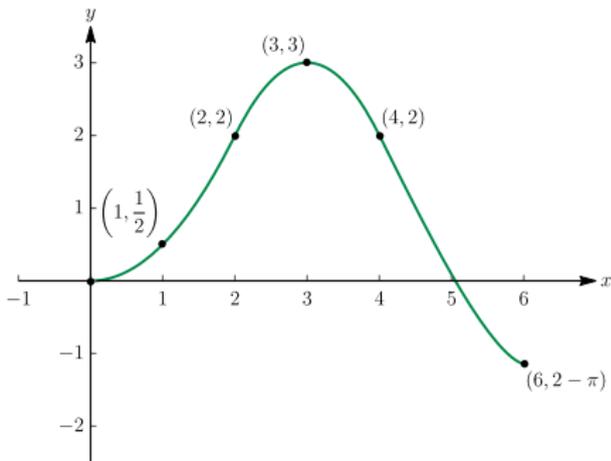
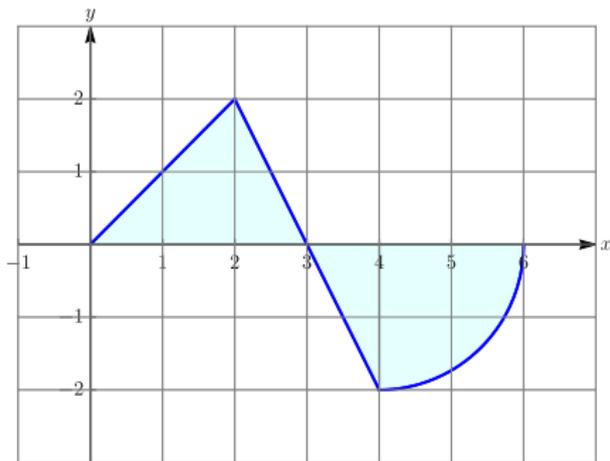
$$g(3) = \int_0^3 f(t) dt = 2 + \frac{1}{2} \cdot 1 \cdot 2 = 3$$

$$g(4) = \int_0^4 f(t) dt = 3 - \frac{1}{2} \cdot 1 \cdot 2 = 2$$

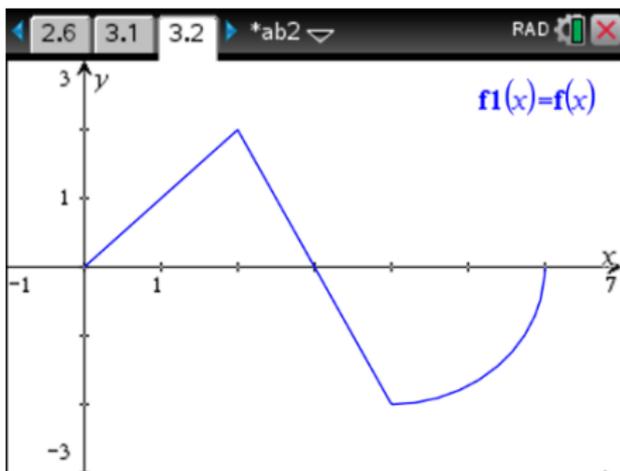
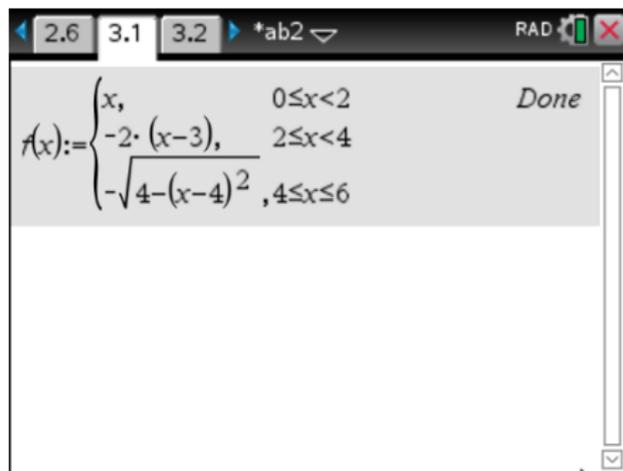


$$g(6) = \int_0^6 f(t) dt = 2 - \frac{1}{4} \cdot \pi \cdot 2^2 = 2 - \pi$$

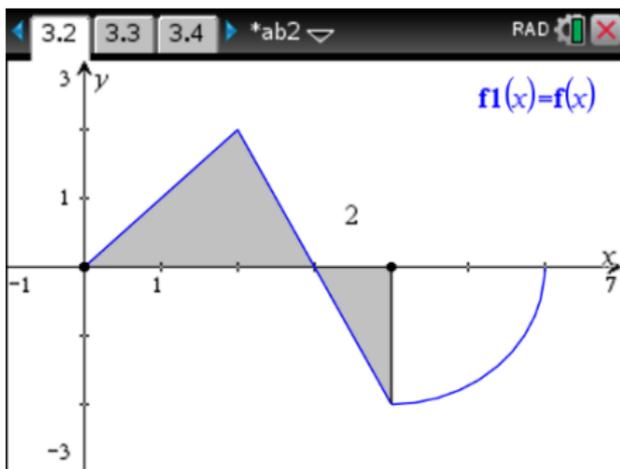
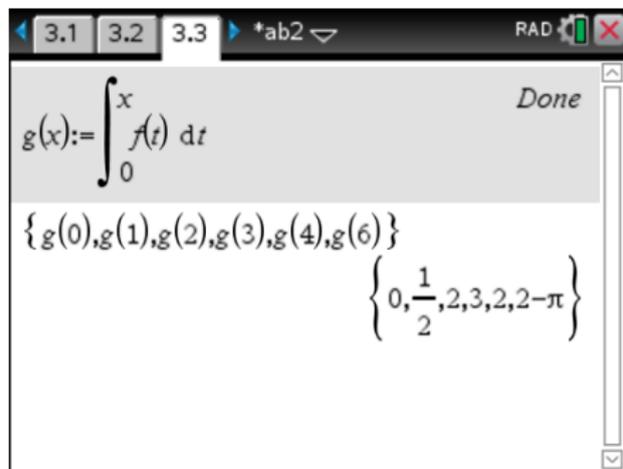
Plot the points found in part (a), and connect with a smooth curve.



Technology Solution

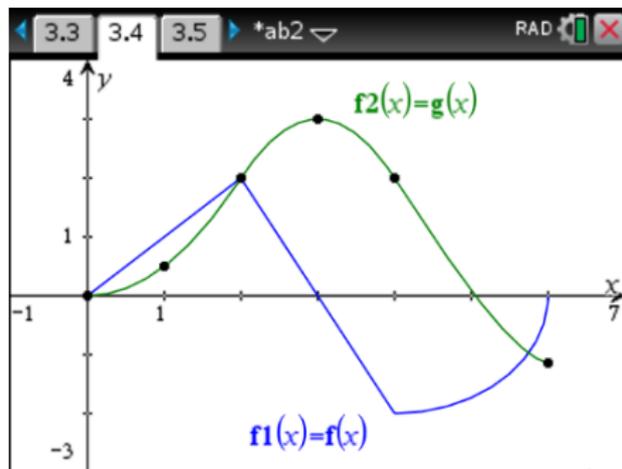


Technology Solution



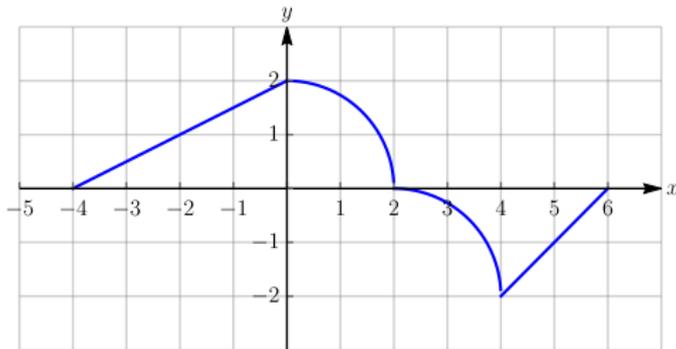
Technology Solution

The graphs of f and g on the same coordinate axes.



Example 2 Use the Graph of f'

Let f be a function defined on the closed interval $-4 \leq x \leq 6$ with $f(0) = -2$. The graph of f' , the derivative of f , consists of two line segments and two quarter circles as shown in the figure.



- On what intervals, if any, is f increasing? Justify your answer.
- Find the x -coordinate of each point of inflection on the graph of f in the open interval $-4 < x < 6$. Justify your answer.
- Find an equation for the line tangent to the graph of f at the point $(0, -2)$.
- Find $f(-4)$ and $f(4)$.

Example 3 The Silo Effect

The rate at which wheat is flowing into a silo on a farm in Kansas is modeled by the function R , where $R(t) = 6.1 + 5 \sin\left(\frac{t^2}{15}\right)$ m³/h, t is measured in hours, and $0 \leq t \leq 12$. Wheat flows from the silo to grain trucks at a rate modeled by $G(t) = 4.5te^{-0.2t}$ m³/h, for $0 \leq t \leq 12$. There are initially 10 m³ of wheat in the silo at time $t = 0$.

- How many cubic meters of wheat flow into the silo during the 4-hour time interval $0 \leq t \leq 4$?
- Is the amount of wheat in the silo increasing or decreasing at time $t = 8$ hours? Give a reason for your answer.
- At what time t , $0 \leq t \leq 12$, is the amount of wheat in the silo a maximum? Justify your answer.

Solution

- (a) We have the rate at which wheat is flowing into the silo.
We need to accumulate the amount of wheat stored.

$$\int_0^4 R(t) dt = \int_0^4 \left[6.1 + 5 \sin \left(\frac{t^2}{15} \right) \right] dt = 30.954 \text{ m}^3$$

A screenshot of a TI-84 Plus calculator interface. The top status bar shows the mode set to 'RAD' and the cursor is on the '4.2' menu item. The main display area shows the function definition $r(t) := 6.1 + 5 \cdot \sin\left(\frac{t^2}{15}\right)$ on the left and the word 'Done' on the right. Below this, the definite integral $\int_0^4 r(t) dt$ is shown on the left, and the numerical result '30.954' is shown on the right. The calculator interface includes a scroll bar on the right side.

Solution

(b) Consider the difference in rates at time $t = 8$ hours.

$$R(8) - G(8) = -5.680 < 0$$

Since $R(8) < G(8)$, the amount of wheat in the silo is decreasing at time $t = 8$ hours.

A screenshot of a TI-84 Plus calculator interface. The top status bar shows the window number 4.3, the current page 4.4, the next page 4.5, and the mode *ab2. The mode indicator shows RAD (radians) and a green battery icon. The main display area shows two lines of text: the first line is the function definition $g(t) := 4.5 \cdot t \cdot e^{-0.2 \cdot t}$ followed by the word "Done", and the second line shows the evaluation $r(8) - g(8)$ resulting in -5.680 . A vertical scroll bar is visible on the right side of the display area.

Solution

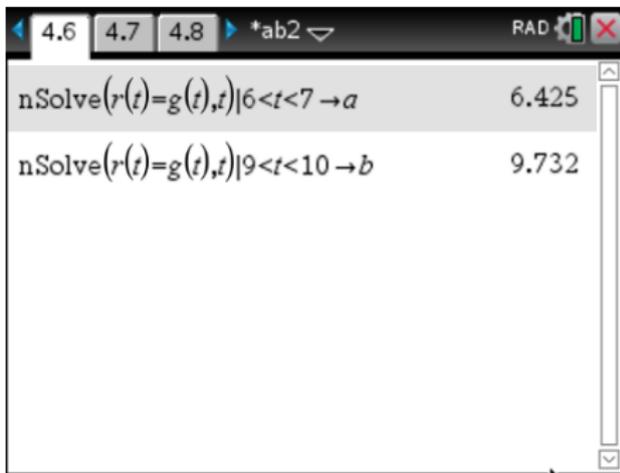
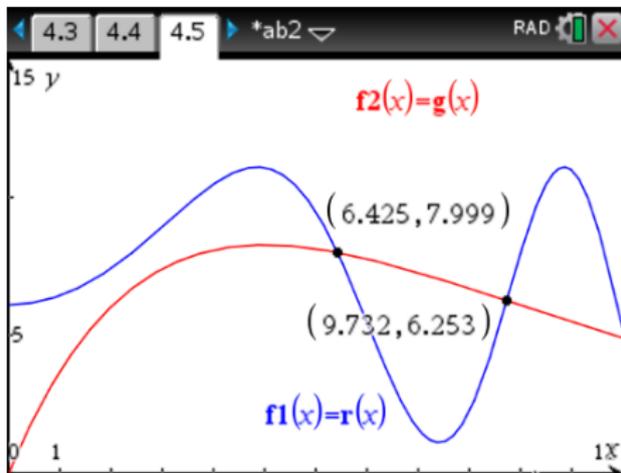
(c) The amount of wheat in the silo at time t , $0 \leq t \leq 12$, is

$$A(t) = 10 + \int_0^t [R(x) - G(x)] dx$$

Find the derivative and the critical numbers.

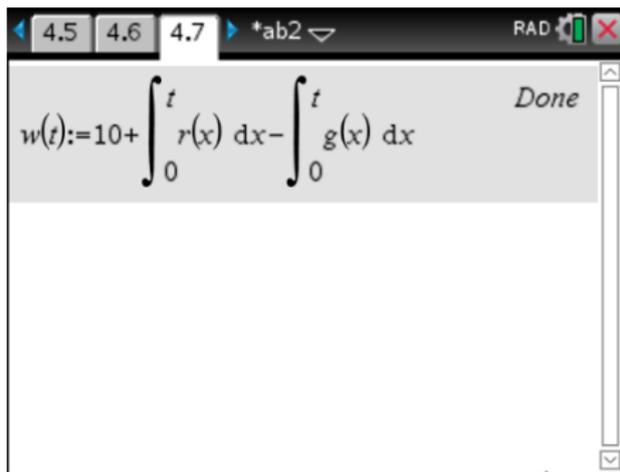
$$A'(t) = R(t) - G(t)$$

$$R(t) - G(t) = 0 \implies t = 6.425, 9.732$$



Solution

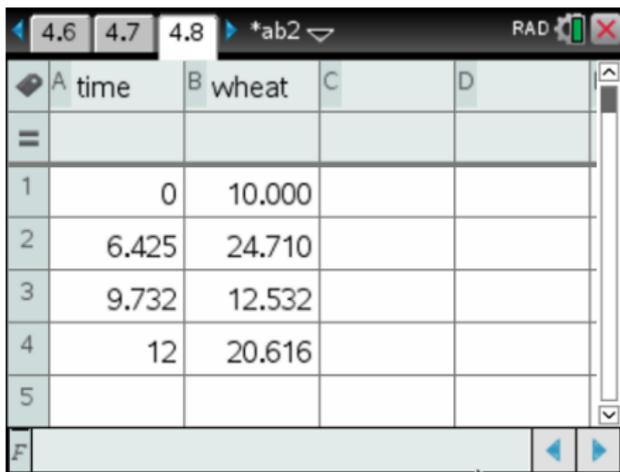
(c) Create a table of values for all relevant critical numbers and endpoints.



TI-84 Plus calculator screen showing the definition of a function $w(t)$ as the difference of two integrals:

$$w(t) := 10 + \int_0^t r(x) \, dx - \int_0^t g(x) \, dx$$

The screen also shows the mode set to RAD and the word "Done" in the top right corner.



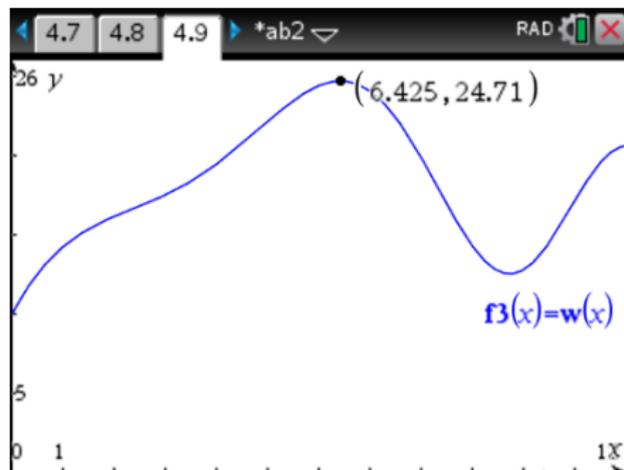
TI-84 Plus calculator screen showing a table of values for time and wheat:

A	time	B	wheat	C	D
=					
1	0	10.000			
2	6.425	24.710			
3	9.732	12.532			
4	12	20.616			
5					

The amount of wheat in the silo is a maximum at time $t = 6.425$ hours.

Solution

Here is a graph of the amount of wheat in the silo at time t .



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