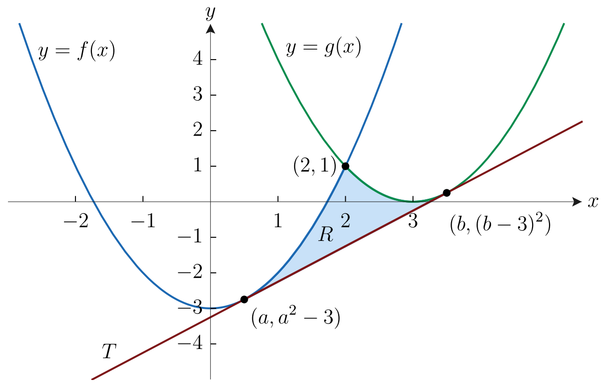
Monday Night Calculus (in the afternoon), April 11, 2022

1. The region R is bounded by the graph of of $f(x) = x^2 - 3$, $g(x) = (x-3)^2$, and the line T, as shown in the figure. The line T is tangent to the graph of f at the point $(a, a^2 - 3)$ and tangent to the graph of g at the point $(b, (b-3)^2)$.



(a) Show that $a = b - 3$.

$$f'(x) = 2x \quad \text{and} \quad g'(x) = 2(x-3) = 2x - 6$$

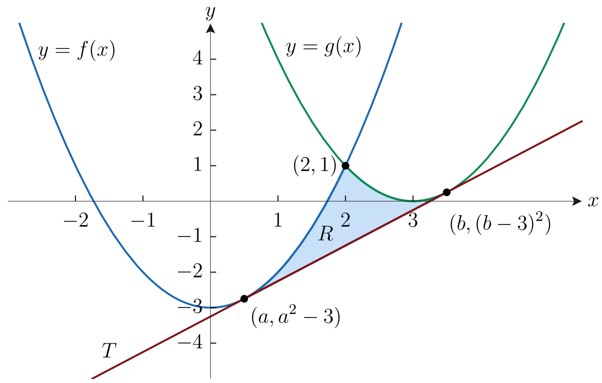
$$f'(a) = g'(b) \;\Rightarrow\; 2a = 2b - 6 \;\Rightarrow\; a = b - 3$$

(b) Find the numerical value of a and the numerical value of b.

$$\text{Slope of } T = \frac{(b-3)^2 - (a^2 - 3)}{b - a}$$

$$g'(b) = 2b - 6 = \frac{(b-3)^2 - (a^2-3)}{b-a} = \frac{(b-3)^2 - ((b-3)^2 - 3)}{b - (b-3)} = \frac{3}{3} = 1$$

$$2b - 6 = 1 \;\Rightarrow\; 2b = 7 \;\Rightarrow\; b = \frac{7}{2} \;\Rightarrow\; a = \frac{7}{2} - 3 = \frac{1}{2}$$



(c) Write an equation of the line T.

Point: $\left(\frac{1}{2}, \frac{1}{4} - 3\right) = \left(\frac{1}{2}, -\frac{11}{4}\right)$ Slope: $f'\left(\frac{1}{2}\right) = 2 \cdot \frac{1}{2} = 1$

An equation for the tangent line T is

$$y + \frac{11}{4} = 1\left(x - \frac{1}{2}\right) \quad \text{or} \quad y = x - \frac{13}{4}$$

(d) Find the area of the region R.

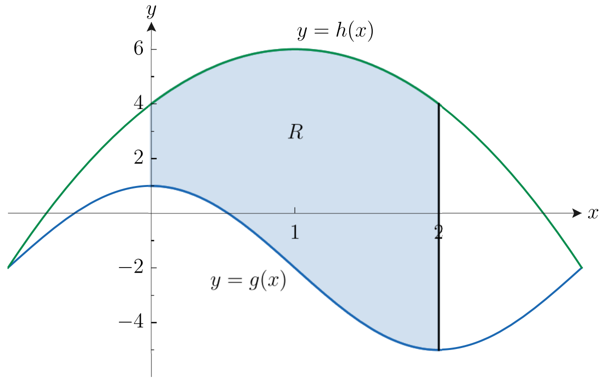
$$A = \int_{1/2}^{2} [f(x) - T(x)]\,dx + \int_{2}^{7/2} [g(x) - T(x)]\,dx$$

$$= \frac{9}{8} + \frac{9}{8} = \frac{9}{4}$$

(e) Challenge: The region R is revolved about the line $y = -4$. Find the volume of the resulting solid.

2. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure.

(Mark Wilson)



(a) Find the area of R.

$$A = \int_0^2 [h(x) - g(x)]\,dx$$

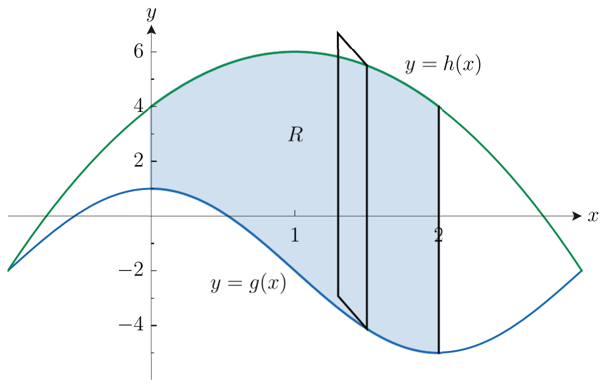
$$= \int_0^2 \left[(6 - 2(x-1)^2) - \left(-2 + 3\cos\left(\frac{\pi}{2}x\right)\right)\right] dx$$

$$= \left[6x - \frac{2}{3}(x-1)^3 + 2x - \frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right]_0^2$$

$$= \left[\left(12 - \frac{2}{3} + 4 - 0\right) - \left(0 + \frac{2}{3} + 0 + 0\right)\right]$$

$$= 16 - \frac{4}{3} = \frac{44}{3}$$

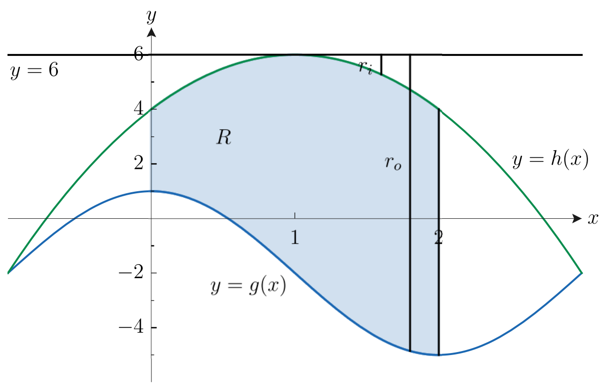
(b) The base of a solid S is the region R. Cross-sections of S perpendicular to the x-axis are rectangles with height $1/3$ the length of the base. Find the volume of S.



$$A(x) = (h(x) - g(x)) \cdot \frac{1}{3}(h(x) - g(x)) = \frac{1}{3}[h(x) - g(x)]^2$$

$$V = \frac{1}{3}\int_0^2 [h(x) - g(x)]^2\, dx = \frac{1759}{45} = 39.089$$

(c) The region R is revolved about the line $y = 6$. Find the volume of the resulting solid.



$$V = \pi \int_0^2 \left[(6 - g(x))^2 - (6 - h(x))^2\right] dx = \frac{677\pi}{5} = 425.372$$

3. $\lim_{x\to 0} \dfrac{\tan(2x)}{3x\cos 4x}$ (Cheryl Roberson)

Solution

$\lim_{x\to 0} \tan 2x = 0$

$\lim_{x\to 0} (3x\cos 4x) = 0 \cdot 1 = 0$

The limit is in the indeterminate form $\dfrac{0}{0}$. We can use L'Hospital's Rule.

$$\lim_{x\to 0} \frac{\tan(2x)}{3x\cos 4x} = \lim_{x\to 0} \frac{\sec^2(2x)\cdot 2}{3\cos 4x + 3x\cdot 4(-\sin 4x)}$$
$$= \frac{1\cdot 2}{3+0} = \frac{2}{3}$$

Solution (Mark Kiraly)

$$\lim_{x\to 0} \frac{\tan(2x)}{3x\cos 4x} = \lim_{x\to 0} \left[\frac{1}{3x}\cdot\frac{\sin 2x}{\cos 2x}\cdot\frac{1}{\cos 4x}\right]$$
$$= \lim_{x\to 0} \left[\frac{1}{3}\cdot\frac{2}{2x}\cdot\frac{\sin 2x}{\cos 2x}\cdot\frac{1}{\cos 4x}\right]$$
$$= \lim_{x\to 0} \left[\frac{1}{3}\cdot 2\cdot\frac{\sin 2x}{2x}\cdot\frac{1}{\cos 2x}\cdot\frac{1}{\cos 4x}\right]$$
$$= \frac{1}{3}\cdot 2\cdot 1\cdot 1\cdot 1 = \frac{2}{3}$$

4. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+6} n}{n^2 + 9}$$

Solution

$$\left| \frac{(-1)^{n+6} n}{n^2 + 9} \right| = \frac{n}{n^2 + 9}$$

Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n \to \infty} \frac{\frac{n}{n^2 + 9}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{n^2 + 9} \cdot \frac{n}{1}$$

$$= \lim_{n \to \infty} \frac{n^2}{n^2 + 9} = \lim_{n \to \infty} \frac{1}{1 + \frac{9}{n^2}} = 1$$

Therefore, the series is not absolutely convergent.

Consider the function $f(n) = \dfrac{n}{n^2 + 9}$

$$f'(n) = \frac{(n^2 + 9) \cdot 1 - n \cdot 2n}{(n^2 + 9)^2} = \frac{9 - n^2}{(n^2 + 9)^2}$$

$$\lim_{n \to \infty} \frac{n}{n^2 + 9} = \lim_{n \to \infty} \frac{1}{n + \frac{9}{n}} = 0$$

Therefore, the series is conditionally convergent.

5. Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+1} \right)^n$$

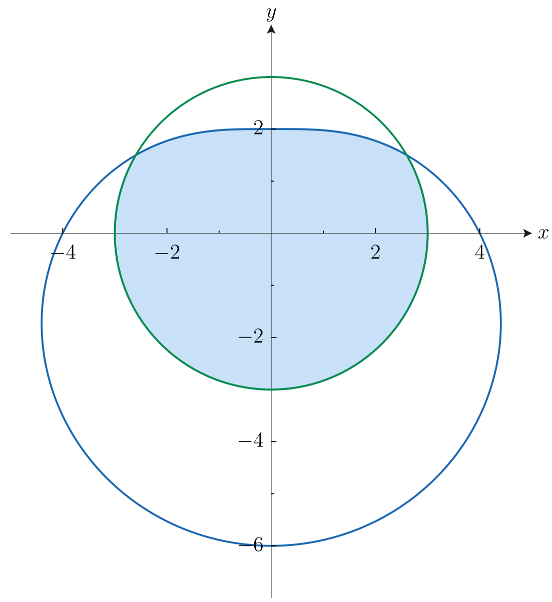
Solution

Use the Root Test.

$$\lim_{n \to \infty} \sqrt[n]{\left| \left(\frac{n+1}{3n+1} \right)^n \right|} = \lim_{n \to \infty} \frac{n+1}{3n+1} = \frac{1}{3} < 1$$

The series is absolutely convergent, and therefore convergent.

6. The graphs of the polar equations $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure.



The graphs intersect when $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{5\pi}{6}$.

(a) Let R be the region that lies inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$, as shaded in the figure. Find the area of R.

$$A = \frac{9\pi}{2} + 2\int_0^{\pi/6} \frac{1}{2} \cdot 3^2 \, d\theta + 2\int_{\pi/6}^{\pi/2} \frac{1}{2}(4 - 2\sin\theta)^2 \, d\theta$$

$$= \frac{9\pi}{2} + \frac{3\pi}{2} + 6\pi - \frac{15\sqrt{3}}{2}$$

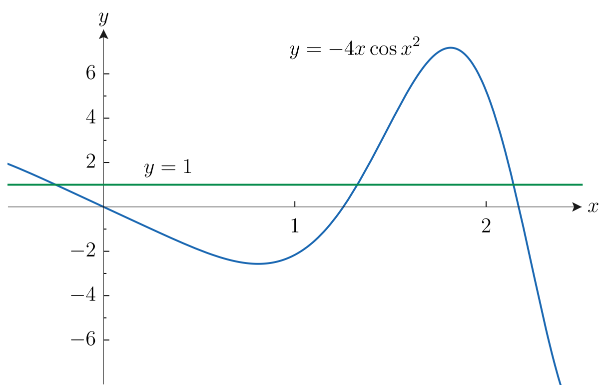
$$= 12\pi - \frac{15\sqrt{3}}{2}$$

(b) A particle is traveling along the polar curve $r = 4 - 2\sin\theta$ so that at time t, $\theta = t^2$ for $1 \le t \le 2$. Find the value of t when $\dfrac{dr}{dt} = 1$.

$$r = 4 - 2\sin\theta \quad \Rightarrow \quad \frac{dr}{d\theta} = -2\cos\theta$$

$$\theta = t^2 \quad \Rightarrow \quad \frac{d\theta}{dt} = 2t$$

$$1 = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = (-2\cos\theta)(2t) = -4t\cos t^2 \quad \Rightarrow \quad t = 1.327$$



(c) For the particle described in part (b), find the time t, in the interval $1 \le t \le 2$, for which the x-coordinate of its position is -1. Find the velocity of the particle at this time.

$$x(t) = r\cos t = (4 - 2\sin t^2)\cos t$$

$$x(t) = -1 = (4 - 2\sin t^2)\cos t \quad \Rightarrow \quad t = 1.808$$

$$x(t) = (4 - 2\sin t^2)\cos t \quad \Rightarrow \quad x'(t) = (-4t\cos t^2)\cos t + (4 - 2\sin t^2)(-\sin t)$$

$$y(t) = r\sin t = (4 - \sin t^2)\sin t \quad \Rightarrow \quad y'(t) = (-4t\cos t^2)\sin t + (4 - 2\sin t^2)\cos t$$

$$\mathbf{v}(1.808) = \langle x'(1.808), y'(1.808)\rangle = \langle -5.821, 5.973\rangle$$